Generalizations of Gödel's second theorem to systems of arithmetic based on non-classical logic

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Basic questions

- What is ‘the right’ formulation of Gödel’s second incompleteness theorem?
- What is its most general formulation?

Generalizations of Gödel's second incompleteness theorem

Various assumptions involved in Gödel's theorem can be relaxed:

- One can weaken the axioms of arithmetic (see Shepherdson, Pudlák, Visser);
- One can weaken the requirements on the proof predicate (see Kreisel, Feferman, Löb, Jeroslow);
- One can consider theories modulo interpretability (Feferman, Friedman, Visser);
- One can weaken the logic.

It is this latter aspect that we are going to explore.

Two well-known examples:

- intuitionistic arithmetic HA;
- equational (say, primitive recursive) arithmetic.

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Comparison with Gödel's First

Gödel's second theorem (G2) is more problematic than the first one (G1):

- G1 is well understood in the context of recursion/computability theory;
- There are abstract logic-free formulations due to Kleene ('symmetric form'), Smullyan ('representation systems'), and others.
- G2 has more to do with the modal-logical properties of the provability predicate and the self-referentiality.

Problems with G2

- The main problem with G2 is that we cannot easily delineate a class of formulas that 'mean' consistency.
- A lucky circumstance is that G2 also holds for larger syntactically defined classes of formulas, some of which are intensionally correct (adequately express consistency), but some are not.
- G2 holds for all provability predicates defined by Σ_1-numerations (Feferman).

A Σ_1-formula $\alpha(x)$ (numeration) defines the set of axioms of T.
A provability formula $\mathsf{Prov}_\alpha(x)$ is determined by α.

Derivability conditions

Hilbert–Bernays–Löb derivability conditions can be considered as stating axiomatically the minimal requirements for a natural provability predicate for a given theory T within S:

- $S \vdash \varphi \quad \Rightarrow \quad S \vdash \Box\varphi$;
- $S \vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- $S \vdash \Box\varphi \to \Box\Box\varphi$.

They do not define naturality, but suffice for G2.

Abstract formulations of G2

Theorem

Suppose S contains classical propositional calculus, $\Box$ satisfies Löb's conditions in S and S is consistent.

If there is a p such that $S \vdash p \leftrightarrow \neg\Box p$, then $S \nvdash \neg\Box\bot$.

Usually, the existence of a fixed point is guaranteed by a substitution function being definable in S. This, in turn, presupposes that S has some minimal arithmetic in it, such as Robinson's system Q. (The substitution function comes in one package with all other computable functions.)

Q is much weaker than what is usually required to prove Löb's conditions in S (some induction is needed).

Abstract consequence relations

Def. An *abstract consequence relation* is a tuple $S = (L_S, \leq_S, \top, \bot)$ where

- L_S is a set (called the set of *sentences* of S);
- $\leq_S$ is a reflexive, transitive relation on L_S;
- $\top$ '*axiom*' and $\bot$ '*contradiction*' are elements of L_S.

Then we can define:

- x is *provable in* S if $\top \leq_S x$
- x is *refutable in* S if $x \leq_S \bot$
- $x =_S y$ if $x \leq_S y$ and $y \leq_S x$.

Remarks

- We do not assume either $\bot \leq_S x$ or $x \leq_S \top$. Nor do we assume the existence of any logical connectives (such as negation) in S.
- S is called *inconsistent* if $\top \leq_S \bot$. If S is consistent then no sentence is both provable and refutable.
- S is called r.e., if so is $\leq_S$.
- If S is r.e., consistent and complete, then S is decidable.

Provability and refutability internalized

We introduce two operators $\Box, \boxtimes : L_S \to L_S$ representing provability and refutability predicates in S.

We assume the following conditions (omitting the subscript $_S$):

1. $x \leq y \;\Rightarrow\; \Box x \leq \Box y,\ \boxtimes y \leq \boxtimes x.$
2. $\top \leq \boxtimes \bot;$
3. $x \leq \Box y,\ x \leq \boxtimes y \;\Rightarrow x \leq \boxtimes \top;$
4. $\boxtimes x \leq \Box \boxtimes x.$

$(L_S, \leq_S, \top, \bot, \Box, \boxtimes)$ is called an *abstract provability structure* (APS).

Abstract version of G2

It seems better in this context to use a Jeroslow-type rather than a Gödel-type fixed point: p says 'p *is refutable*.'

Theorem

Let S be an APS such that there is a $p =_S \boxtimes p$. Then:

1. If S is consistent then $\boxtimes\top$ is irrefutable in S;
2. Statement 1 is formalizable in S: $\boxtimes \boxtimes \top \leq_S \boxtimes\top$.

Proofs

Let $p = \boxtimes p$. First we prove formalized G2:

1. $p = \boxtimes p \leq \Box \boxtimes p \leq \Box p$
2. $p \leq \boxtimes p$, hence
3. $p \leq \boxtimes \top$
4. $\boxtimes \boxtimes \top \leq \boxtimes p = p \leq \boxtimes \top$ (formalized G2)

Proof of nonformalized G2:

1. Assume $\boxtimes \top \leq \bot$
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Adding an implication

We go to a more familiar format $\Gamma \vdash \varphi$ where Γ is a finite multiset and φ an element of a given set L. Implication is undertood as a binary operation on L.

Def. A consequence relation with a *good implication on* L is a structure $S = (L, \vdash, \to, \top, \bot)$ such that

1. $\varphi \vdash \varphi$;
2. if $\Gamma, \psi \vdash \varphi$ and $\Delta \vdash \psi$ then $\Gamma, \Delta \vdash \varphi$;
3. $\Gamma, \varphi \vdash \psi \iff \Gamma \vdash (\varphi \to \psi)$.

Remark. Setting $\varphi \leq \psi$ as $\varphi \vdash \psi$ yields an abstract consequence relation in the previous sense.

Remarks

For any consequence relation with a good implication:

- Both of the 'multiplicative' implication rules hold:

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \qquad \frac{\Gamma \vdash \varphi \quad \Delta, \psi \vdash \theta}{\Gamma, \Delta, \varphi \to \psi \vdash \theta}$$

- Implication respects deductive equivalence;
- One can introduce a negation $\neg\varphi := (\varphi \to \bot)$, for which the contraposition rule is derivable.

Löb's conditions in a weak context

For consequence relations with a good implication, Löb's conditions can be stated literally.

Question: Suppose a consequence relation with a good implication S has an operator $\Box$ satisfying Löb's conditions. Does G2 hold?

Löb's conditions are equivalent to:

1. $\Gamma \vdash \varphi$ implies $\Box\Gamma \vdash \Box\varphi$;
2. $\Box\varphi \vdash \Box\Box\varphi$.

Apparently contraction is needed

- Assuming Löb's conditions we can define refutability: $\boxtimes\varphi := \Box\neg\varphi$.
- All conditions of APS are then satisfied except for

$$\varphi \vdash \Box\psi, \quad \varphi \vdash \boxtimes\psi \;\Rightarrow\; \varphi \vdash \boxtimes\top,$$

which has a hidden contraction on the left.

Conjecture. There is a consequence relation with a good implication and with a $\Box$ satisfying Löb's conditions such that the abstract version of G2 does not hold.

Does it hold for any of the 'mathematical' axiomatic systems?

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Context: Grishin's work on set theory without contraction

- Vyacheslav Grishin studied set theory based on a logic without contraction in the 70s and 80s. That's how the first order affine logic (without exponentials) was actually introduced for the first time.
- He proved, however, that the extensionality principle allows this system to actually *prove* contraction. In particular, full comprehension is not working well with extensionality even if there is no postulated contraction in the logic.
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- But what about arithmetic without contraction?

Arithmetic without contraction

(Daniyar Shamkanov, unwritten)

Logic: the usual affine predicate logic without exponentials (with both multiplicatives and additives).
Sequents: Tait-style with multisets of formulas understood as a multiplicative disjunction, negation defined via de Morgan laws.

Axioms: $\Gamma, \neg\varphi, \varphi$; $\quad \Gamma, \top$
Rules:

$$\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi} \qquad \frac{\Gamma, \varphi \quad \Delta, \psi}{\Gamma, \Delta, \varphi \otimes \psi} \qquad \frac{\Gamma, \varphi_i}{\Gamma, \varphi_1 \vee \varphi_2} \qquad \frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \oplus \psi} \qquad \frac{\Gamma}{\Gamma, \bot}$$

$$\frac{\Gamma, \varphi(a)}{\Gamma, \forall x\, \varphi(x)} \qquad \frac{\Gamma, \varphi(t)}{\Gamma, \exists x\, \varphi(x)} \qquad \frac{\Gamma, \varphi \quad \Delta, \neg\varphi}{\Gamma, \Delta} \text{ (Cut)}$$

Arithmetical axioms

Axioms:

1. $\neg Sx = 0, \quad Sx = Sy \leftrightarrow x = y;$
2. $x + 0 = x, \quad x + Sy = S(x + y);$
3. $x \cdot 0 = 0, \quad x \cdot Sy = x \cdot y + x.$

Induction rule:

$$\frac{\varphi(0), \quad \forall x\, (\varphi(x) \to \varphi(Sx))}{\forall x\, \varphi(x)}$$

Remarks.

1. No equality axioms. Induction as a rule.
2. Axioms can be used any number of times in derivations.

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Some features

- The equality schema is, in fact, provable:

$$\vdash x = y \rightarrow (\varphi(x, \vec{a}) \rightarrow \varphi(y, \vec{a})),$$

  for any formula $\varphi(x, \vec{a})$.
- Postulating induction as a *schema* yields contraction and the system becomes equivalent to PA.
- With the induction principle stated as an inference rule we conjecture that the contraction rule is not admissible.

Restricted contraction

We can prove contraction for a restricted class of formulas.

1. If $\vdash A \vee \neg A$, then $\vdash (A \oplus A) \to A$ and $\vdash A \to (A \otimes A)$ (contraction holds for A).
2. If $A \in \Delta_0$ (bounded quantifiers) then $\vdash A \vee \neg A$, and hence contraction holds for A.
3. The induction *schema* for Δ_0-formulas is provable.
4. The substitution function is representable and the fixed-point lemma holds.
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G2 for affine arithmetic

Theorem

G2 holds for the affine arithmetic.

Contraction for Σ_1-formulas is sufficient. We actually use:

$$\boxtimes x \leq \boxtimes y, \quad \boxtimes x \leq \Box y \;\Rightarrow\; \boxtimes x \leq \boxtimes \top.$$

Then the argument goes as before.

In other words, the affine arithmetic validates the rule:

$$\frac{\Gamma, \Box\varphi, \Box\varphi \vdash \psi}{\Gamma, \Box\varphi \vdash \psi}.$$

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Open questions about affine arithmetic

1. Prove that in AA contraction is not admissible;
2. Disjunction property: $\vdash A \vee B$ implies $\vdash A$ or $\vdash B$?
3. Numerical existence property?
4. Provably recursive functions?
5. Conservation results for classical arithmetics?
6. Translations? Realizability?

Open questions about G2

1. Does there exist a reasonable arithmetic based on a non-classical logic for which G2 *fails*?

   The example of AA shows that arithmetical axioms may have the effect on logic making part of it classical. So, we may need to restrict the arithmetical part as well.

2. Can we replace the condition of the existence of fixed points by something more natural on the same level of abstraction in the abstract formulation of G2?

Some related work

1. V.N. Grishin (1982): Predicate and set-theoretic calculi based on logic without contractions.
2. U. Petersen (2000): Logic without contraction as based on inclusion and unrestricted abstraction.
3. K. Terui (2004): Light affine set theory: a naive set theory of polynomial time.
4. R. McKinley (2008): Soft linear set theory.
5. G. Japaridze (2011): Introduction to Clarithmetic.