Flexible formulae of arithmetic

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June 18, 2014
JAF33

Theorem (Kripke, 1962)

If T is an r.e., consistent extension of Q, then there is a Σ_n-formula $\gamma(x)$ that is flexible for Σ_n in the following sense: For each Σ_n-formula $\delta(x)$, the theory $T + \forall x(\gamma(x) \leftrightarrow \delta(x))$ is consistent.

Theorem (Model-theoretic version)

If T is an r.e., consistent extension of Q, then there is a Σ_n-formula such that $\mathbb{N} \models \forall x \neg\gamma(x)$, and for each Σ_n-formula $\delta(x)$, there is a model $\mathcal{M} \models T$ such that $\mathcal{M} \models \forall x(\gamma(x) \leftrightarrow \delta(x))$.

Theorem (Woodin, 2011)

There is a Σ_1*-formula* $\psi(x)$ *such that*

1. $\mathbb{N} \models \forall x \neg\psi(x)$,
2. *for each countable model* $\mathcal{M}$ *of PA, and for each* $\delta(x)$ *such that* $\mathcal{M} \models \forall x(\psi(x) \to \delta(x))$*, and the* $\mathcal{M}$*-extension of* $\delta(x)$ *is* $\mathcal{M}$*-finite, there is an end-extension* $\mathcal{N} \models PA$ *of* $\mathcal{M}$ *such that* $\mathcal{N} \models \forall x(\psi(x) \leftrightarrow \delta(x))$.

Theorem (Enayat-Shavrukov, 2012, unpublished)

There is a Σ_1-formula $\psi(x)$ such that

1. $\mathbb{N} \models \forall x \neg\psi(x)$,
2. *for each ~~countable~~ model $\mathcal{M}$ of PA, and for each $\delta(x)$ such that $\mathcal{M} \models \forall x(\psi(x) \rightarrow \delta(x))$, and the $\mathcal{M}$-extension of $\delta(x)$ is $\mathcal{M}$-finite, there is an end-extension $\mathcal{N} \models$ PA of $\mathcal{M}$ such that $\mathcal{N} \models \forall x(\psi(x) \leftrightarrow \delta(x))$.*

- T is an r.e., consistent extension of $\mathrm{I}\Sigma_1$.
- Sat_{Σ_n} is a satisfaction predicate for (binary) Σ_n-formulae, such that $T \vdash \mathrm{Sat}_{\Sigma_n}(\ulcorner \phi \urcorner, x, y) \leftrightarrow \phi(x, y)$.
- $\mathrm{Sel}\{\phi\}$ is a formula minimising the witness for ϕ, so that $T \vdash \exists z \phi(z) \rightarrow \exists! z \, \mathrm{Sel}\{\phi\}(z)$.
- The ith (unary) partial recursive function f_i is the function with graph defined by $\mathrm{Sat}_{\Sigma_1}(i, x, y)$.
- $R(x, y, z)$ is the formula $\mathrm{Sel}\{\mathrm{Sat}_{\Sigma_1}\}(x, y, z)$, which strongly represents a universal machine.

Lemma (Kripke, 1962)

There is an index e such that for each k, the theory $T + R(e, e, k)$ is consistent.

Proof.

Let e be the index of the function f defined by

$$f(n) = k \text{ iff } T \vdash \neg R(n, n, k).$$

Suppose $T \vdash \neg R(e, e, k)$. Then $f(e) = k$, so $T \vdash R(e, e, k)$, contradicting the consistency of T. Hence $T + R(e, e, k)$ is consistent. □

Let $\gamma(x) := \exists z(R(e, e, z) \wedge \mathrm{Sat}_{\Sigma_n}(z, x))$, and let $\delta(x)$ be any Σ_n-formula. $T + R(e, e, \ulcorner\delta\urcorner)$ is consistent, so $T + \forall x(\gamma(x) \leftrightarrow \delta(x))$ is consistent.

Theorem

There is a Σ_n-formula $\gamma(x)$ such that for each model $\mathcal{M}$ of $T + \mathrm{Con}(T)$,

1. $\mathcal{M} \models \forall x \neg \gamma(x)$,
2. *for each Σ_n-formula $\delta(x)$, there is an end-extension $\mathcal{N} \models T$ of $\mathcal{M}$ such that $\mathcal{N} \models \forall x(\gamma(x) \leftrightarrow \delta(x))$.*

Lemma

There is an index e such that

1. $T \vdash \forall y(\mathrm{Con}(T) \to \neg R(e, e, y))$,
2. $T \vdash \forall y(\mathrm{Con}(T) \to \mathrm{Con}(T + R(e, e, y))$.

Let $\gamma(x) := \exists z(R(e, e, z) \wedge \mathrm{Sat}_{\Sigma_n}(z, x))$.

- By (1), if $\mathcal{M} \models \mathrm{Con}(T)$, then $\mathcal{M} \models \forall x \neg\gamma(x)$.
- By (2), if $\mathcal{M} \models \mathrm{Con}(T)$, then $\mathcal{M} \models \mathrm{Con}(T + R(e, e, \ulcorner\delta\urcorner))$.
- By the Low Arithmetised Completeness Theorem, there is an end-extension $\mathcal{N} \models T$ of $\mathcal{M}$ such that $\mathcal{N} \models R(e, e, \ulcorner\delta\urcorner)$.
- Hence $\mathcal{N} \models \forall x(\gamma(x) \leftrightarrow \delta(x))$.

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