Recursive saturation: a valuation theoretic approach

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Algebraic characterizations of κ-saturation of

- **ODAG** = ordered divisible Abelian groups
  (by S. Kuhlmann in 1990)

- **RCF** = real closed fields
  (by F.-V. Kuhlmann, S. Kuhlmann, Marshall, Zekavat 2002)

are known.

Goal: algebraic characterization of recursive saturation for ordered divisible abelian groups and for real closed fields.

Ordered Divisible Abelian Groups

Definition

$(G, +, 0, <)$ is an ordered divisible Abelian group if $(G, +, 0)$ is a divisible Abelian group and $<$ is compatible with $+$

Examples: $(\mathbb{Q}, +, 0, <)$, any ordered $\mathbb{Q}$-vector space.

ODAG admits elimination of quantifiers in the language of ordered groups.

Valuation theory notions: ODAG

Let $(G, +, 0, <)$ be a divisible ordered Abelian group.
$|x| = \max\{-x, x\}$ for all $x \in G$. For nonzero $x, y \in G$ we define

$$x \sim y \quad \text{if there exist } n \in \mathbb{N} \quad \text{s.t. } n|x| \geq |y| \quad \text{and} \quad n|y| \geq |x|.$$

$\sim$ is an equivalence relation. The quotient set
$\Gamma = G - \{0\}/\sim = \{[x] : x \in G - \{0\}\}$ is the value set of G.

Γ is endowed with a linear order defined as

$$[x] < [y] \quad \text{iff} \quad n|y| < |x| \quad \text{for all} \quad n \in \mathbb{N}$$

The map $v \colon G \to \Gamma \cup \{\infty\}$ defined by $v(0) := \infty$ and $v(x) = [x]$ if $x \neq 0$ is a valuation on G, i.e., for every $x, y \in G$, $v(x) = \infty$ if and only if $x = 0$, $v(nx) = v(x)$ for all $n \in \mathbb{Z}^{\times}$, and $v(x + y) \geq \min\{v(x), v(y)\}$. v is the natural valuation on G

Valuation theory notions: ODAG

Definition

Let $[x]$ be in the value set Γ. The *archimedean component associated to* $[x]$, $A_{[x]}$, is the maximal archimedean subgroup of G containing x.

Calculating $A_{[x]}$

If $y \in [x]$ for $[x] \in \Gamma$, we set

$$\frac{y}{x} = \sup\{r \in \mathbb{Q} \mid rx < y\} \text{ and}$$
$$A_{[x],x} = \left\{\frac{y}{x} \mid y \in [x]\right\} \cup \{0\} \subseteq (\mathbb{R}, +, 0, <).$$

Then, $A_{[x]} \cong A_{[x],x}$.

Example Let $G = \{a + bx \mid a \in \mathbb{Q} \; \& \; b \in \mathbb{R}\}$. Then

$(G, +, 0 <)$, where $x >> \mathbb{R}$ is a DOAG.

The value set is:

$$\Gamma = \{[1], [x]\}$$

The Archimedean components are:

$$A_{[1]} \cong \mathbb{Q} \quad \& \quad A_{[x]} \cong \mathbb{R}$$

Theorem (S. Kuhlmann 1990)

A divisible ordered abelian group G is $\aleph_0$-saturated iff

(i) *the value set Γ of G is a dense linear order without endpoints, and*

(ii) *all archimedean components $A_{[x]}$ of G are isomorphic to $\mathbb{R}$.*

Recursive saturation

Definition

Let L be a computable language and $\mathcal{A}$ an L-structure. $\mathcal{A}$ is *recursively saturated* if for any computable set of L-formulas $\Gamma(\overline{u}, x)$, for all tuples $\overline{a}$ in $\mathcal{A}$ with $|\overline{a}| = |\overline{u}|$, if every finite subset of $\Gamma(\overline{a}, x)$ is satisfied in $\mathcal{A}$, then $\Gamma(\overline{a}, x)$ is satisfied in $\mathcal{A}$.

In other words, a structure $\mathcal{M}$ is recursively saturated if there are no recursive counterexample to ω-saturation.

- $(\mathbb{N}, +, \cdot, 0, 1, <)$ is not recursively saturated because of the type $\{x > n : n \in \mathbb{N}\}$.
- For the same reason $(\mathbb{R}, +, \cdot, 0, 1, <)$ is not recursively saturated.

• For each $\mathcal{A}$ there is $\mathcal{A}^\star$ such that $\mathcal{A} \preceq \mathcal{A}^\star$, $|A| = |A^\star|$, and $\mathcal{A}^\star$ is recursively saturated.

• Any non standard model of Peano Arithmetic has a *certain amount* of recursive saturation: there are no recursive counterexample to ω-saturation of bounded complexity.

Theorem (Barwise-Schlipf)

Suppose $\mathcal{A}$ is countable and recursively saturated. Let Γ be a recursive set of sentences involving some new symbols. If Γ in the language of $\mathcal{A}$ is consistent with $\mathcal{A}$, then $\mathcal{A}$ can be expanded to $\mathcal{A}'$ satisfying Γ. Moreover, $\mathcal{A}'$ can be chosen recursively saturated.

Scott sets

Definition

- $\mathcal{T} \subset 2^{<\omega}$ is a *tree* if $(\forall \sigma \in 2^{<\omega})\ [\sigma \prec \alpha \in \mathcal{T} \implies \sigma \in \mathcal{T}]$.
- $f \in 2^{\omega}$ is a *path* through tree $\mathcal{T}$ if

$$(\forall \sigma \in 2^{<\omega})\ [\sigma \prec f \implies \sigma \in \mathcal{T}].$$

Definition

A nonempty set $S \subset \mathcal{P}(\mathbb{N})$ is a *Scott set* if

1. S is computably closed, i.e., if $r_1, r_2 \in S$ and $r \leq_T r_1 \oplus r_2$ then $r \in S$.
2. If $\mathcal{T} \subset 2^{<\omega}$ is infinite and $\mathcal{T} \leq_T r \in S$, then $\mathcal{T}$ has a path computable in some $r' \in S$.

Scott Sets

Example: Let $\mathcal{A}$ is a nonstandard model of PA, and $a \in \mathcal{A}$,

$$A_a = \{n \in \omega : \mathcal{A} \models p_n | a\}$$

is the subset of $\mathbb{N}$ coded by a in $\mathcal{A}$. Let $SS(\mathcal{A}) = \{A_a : a \in \mathcal{A}\}$

1. $SS(\mathcal{A})$ is computably closed, i.e., if $r_1, r_2 \in SS(\mathcal{A})$ and $r \leq_T r_1 \oplus r_2$ for some $r \in \mathbb{R}$, then $r \in SS(\mathcal{A})$.
2. for any infinite subtree T of $2^{<\omega}$ s.t. $T \in SS(\mathcal{A})$, there is a path in $SS(\mathcal{A})$

1+2 say that $SS(\mathcal{A})$ is a Scott set.

Can extend the notion of coded set also to a real closed field $\mathcal{R}$.

Proposition

Let $\mathcal{M}$ be a non standard model of PA. Then $\mathcal{M}$ is Σ_n-recursively saturated for each $n \in \mathbb{N}$.

Lemma

Let $\mathcal{A}$ be a nonstandard model of PA.

1. For any tuple $\overline{a}$ in $\mathcal{A}$, and any $n \in \omega$, the Σ_n type of $\overline{a}$ (with no parameters) is in $SS(\mathcal{A})$.
2. For any n, if $\Gamma(\overline{x}, \overline{w})$ is a consistent set of Σ_n-formulas belonging to $SS(\mathcal{A})$ and every finite subset of $\Gamma(\overline{x}, \overline{a})$ is satisfied in $\mathcal{A}$, then $\Gamma(\overline{x}, \overline{a})$ is satisfied in $\mathcal{A}$.

The proofs use partial satisfaction classes, i.e. satisfaction classes for Σ_n-formulas.

Scott sets

Scott characterization. A countable Scott set is the collection of subsets of ω coded in some nonstandard model of Peano Arithmetic.

Knight and Nadel characterization A Scott set of cardinality $\aleph_1$ is the collection of subsets of ω coded in some nonstandard model of Peano Arithmetic.

Under CH, all Scott sets arise as collections of subsets of ω coded in nonstandard models of Peano Arithmetic.

Definition

Let L be a computable language and $S \subset P(\omega)$. An L-structure M is *S-saturated* if

1. every type realized in M is computable from some $s \in S$, and
2. if $\tau(x, \bar{y})$ is computable in some $s \in S$ and $\overline{m}$ is a tuple in M such that $\tau(x, \overline{m})$ is finitely satisfiable in M, then $\tau(x, \overline{m})$ is realized in M.

Scott sets are intimately connected with recursively saturated models in the following sense.

Proposition

Let L be a computable language. An L-structure M is recursively saturated if and only if M is S-saturated for some Scott set S.

Theorem (Harnik & Ressayre, D'A. Kuhlmann & Lange JSL 2014)

A divisible ordered abelian group G is recursively saturated iff

(i) *the value set of G is a dense linear order without endpoints, and*

(ii) *all archimedean components $A_{[x],x}$ of G equal a common Scott set S.*

sketch of the proof: ($\Rightarrow$) (i) Let $g, g' \in G$ such that $g, g' > 0$ and $v(g) < v(g')$. The partial type

$$\beta(x, g, g') = \{ng' < x \mid n \in \mathbb{N}\} \cup \{x < ng \mid n \in \mathbb{N}\}$$

is computable and finitely satisfiable (since $v(g) < v(g')$ and G is divisible). By recursive saturation, there is some $h \in G$ such that h realizes $\beta(x, g, g')$ in G. Hence, $v(g) < v(h) < v(g')$. A similar argument shows that Γ has no greatest or least element.

(ii) Since G is recursively saturated, G is S-saturated for some Scott set S. Let $g \in G - \{0\}$ then $A_{[g],g} = S$. For any real r, the partial type $\delta_r(x, y)$ consisting of the formulas

$$qy < x < q'y \quad \text{for all} \quad q, q' \in \mathbb{Q} \quad \text{where} \quad q < r < q'$$

has the same Turing degree as r.

$$\left\{ \begin{array}{l} G \text{ is } S-\text{saturated} \\ G \text{ is divisible} \\ S \text{ is computably closed} \end{array} \right. \qquad \Rightarrow$$

$r \in S$ iff the type $\delta_r(x, g)$ is realized in G iff $r \in A_{[g],g}$.
Hence, $A_{[g],g} = S$.

A set $\{g_1, \ldots, g_n\} \subset G$ is called *valuation independent* if for all $q_1, \ldots, q_n \in \mathbb{Q}$,

$$v(\sum q_i g_i) = \min\{v(g_i) \mid q_i \neq 0\}.$$

A $\mathbb{Q}$-basis $\{g_1, \ldots, g_n\}$ for G is called a *valuation basis* if it is valuation independent.

Theorem (Brown)

Every valued vector space of countable dimension admits a valuation basis.

($\Leftarrow$) Let $\tau(v, \overline{w})$ be a recursive type, $\overline{g} \in G$, and $\tau(v, \overline{g})$ finitely satisfiable in G. The divisible subgroup generated by $\overline{g}$ has a valuation basis, and this is used to show that there is a completion of $\tau(v, \overline{w}) \cup tp_{\overline{g}}(\overline{w})$ in S which is still finitely satisfiable in G, hence realized in some extension of G.

o-minimality of ODAG implies that the type reduces to a cut.

Remark

Let S be a Scott set and Γ a dense linear order without endpoints. The Hahn group $G = \oplus_\Gamma S$ is an example of a recursively saturated divisible ordered abelian group. Hence, every Scott set S appears as the Archimedean component of a recursively saturated divisible ordered abelian group.

For countable groups, these are the only examples:

Corollary

A countable divisible ordered abelian group G is recursively saturated if and only if G is isomorphic to $\oplus_{\mathbb{Q}} S$ for some countable Scott set S.

Real Closed Fields - RCF

Definition

A real closed field is a model of the theory of the ordered field of real numbers in the language $\mathcal{L} = \{+, \cdot, 0, 1, <\}$.

RCF admits elimination of quantifiers, and it is o-minimal, i.e. the 1-definable (with parameters) sets are finite unions of intervals and points.

Theorem (D'A, Knight and Starchenko)

A countable real closed field R is recursively saturated if and only if it has an integer part which is a model of Peano Arithmetic.

An *integer part* is a discretely ordered subring I such that 1 is the least positive element, and for each $x \in R$, there is some $i \in I$ such that $i \leq x < i + 1$.

Valuation theory notions: RCF

If $(R, +, \cdot, 0, 1, <)$ is a RCF then it has a natural valuation associated which is the natural valuation associated to the divisible ordered abelian group $(R, +, 0, <)$.
On the value set $v(R^{\times}/\sim)$ can define a group structure by setting

$$v(x) + v(y) = v(xy) \text{ for all } x, y \in v(R^{\times}).$$

The value set, equipped with this addition, is a divisible ordered abelian group.

Valuation theory notions: RCF

Definition

- Let λ be an infinite ordinal. A sequence $(a_\rho)_{\rho<\lambda}$ contained in R is *pseudo Cauchy* if for every $\rho < \sigma < \tau < \lambda$

$$v(a_\sigma - a_\rho) \ < \ v(a_\tau - a_\sigma).$$

- Let $S = (a_\rho)_{\rho<\lambda}$ be a pseudo Cauchy sequence. An element $x \in R$ is *pseudo limit* of S if

$$v(x - a_\rho) = v(a_{\rho+1} - a_\rho)$$

for all $\rho < \lambda$.

Note that if $(a_\rho)_{\rho<\lambda}$ is a pseudo Cauchy sequence then for all $\rho < \sigma < \lambda$, $v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho)$.

Valuation theory notions: RCF

$R_v = \{r \in R : v(r) \geq 0\}$ is the valuation ring of R, i.e. the finite elements of R

$\mu_v = \{r \in R : v(r) > 0\}$ is the maximal ideal of R, i.e. the infinitesimal elements of R

$k = R_v/\mu_v$ is the residue field of R, it is an archimedean real closed field, so isomorphic to subfield of $\mathbb{R}$

$\aleph_0$-saturation for RCF

Theorem (F.V. Kuhlmann, S. Kuhlmann, Marshall, Zekavat)

Real closed field R is $\aleph_0$-saturated iff

(i) Value group G is $\aleph_0$-saturated,

(ii) Residue field $k \cong \mathbb{R}$,

(iii) every pseudo Cauchy sequence in any subfield $R' \subset R$ of finite transcendence degree over $\mathbb{Q}$ has a pseudo limit in R.

Question

What kinds of pseudo Cauchy sequences must have pseudo limits in the recursively saturated case?

Recursive saturation for RCFs

Theorem (D'A., Kuhlmann, Lange, JSL 2014)

Let R be a RCF. Then R is recursively saturated iff there is a Scott set S s.t.

(i) G is recursively saturated with archimedean components all equal to S,

(ii) $(k, +, \cdot, 0, 1, <) \cong (S, +, \cdot, 0, 1, <)$,

(iii) every pseudo Cauchy sequence *of length ω* in any subfield $R' \subset R$ of finite transcendence degree over $\mathbb{Q}$ that is *computable in an element of S* has a pseudo limit in R.

(iv) every type realized by some n-tuple $\bar{a}$ in R is computable in an element of S.

Definition

Let $\overline{d} \in R$. A sequence $(a_i)_{i<\omega} \subset RC(\bar{d})$ is *computable in* $r \in \mathbb{R}$ *over* $\overline{d}$ if there is an r-computable sequence of formulas $(\theta_i(x, \bar{y}))_{i<\omega}$ such that $\theta_i(x, \bar{d})$ defines a_i in R.

Condition (iv) does not follow from the other three conditions as witnessed by the following example of D. Marker.

Let S be a countable Scott set. The Hahn group $G = \oplus_{\mathbb{Q}} S$ is recursively saturated. Then, $R = S((G))$ satisfies the first three conditions listed in the above Theorem, but R realizes 2-types of arbitrary complexity. Let $f \in 2^\omega$, and let $a = \sum_{n<\omega} f(n)t^{ng}$ for some $g \in G$ with $g > 0$. Then, f is computable in the complete type $\delta(x, y)$ of (a, g). So, R does not satisfy (iv).

THEOREM (Dolich, Knight, Lange & Marker)

Given a Scott set S there is a recursively saturated real closed field R such that S is the residue field of R.

This is constructed as a union of elementary chain.

Definition

Let L be a computable language and $S \subset P(\omega)$. An L-structure M is *S-saturated* if

1. every type realized in M is computable from some $s \in S$, and
2. if $\tau(x, \bar{y})$ is computable in some $s \in S$ and $\overline{m}$ is a tuple in M such that $\tau(x, \overline{m})$ is finitely satisfiable in M, then $\tau(x, \overline{m})$ is realized in M.

Can always find an elementary extension where a finitely satisfiable type is realized. We have to ensure that when we add the elements realizing the types we do not destroy 1) of S-saturation, i.e. the following lemma is needed

Lemma

Let R_0 be a real closed field and assume that 1) holds, i.e. for every tuple $\overline{a}$ in R_0 $tp_{\overline{a}}(\overline{v}) \in S$. Let $\tau(v, \overline{w}) \in S$ and $\overline{b} \in R_0$. Assume that $\tau(v, \overline{b})$ is finitely satisfiable in R_0, and let a realize $\tau(v, \overline{b})$ in an extension of R_0. Then for all $\overline{c} \in R_0$, $tp_{a,\overline{c}}(v, \overline{u})$ belongs to S.

We want an algorithm with oracles in S for deciding whether $R_0(a)^{rc} \models \varphi(a, \overline{c})$. By o-minimality the set defined by $\varphi(v, \overline{c})$ is finite union of intervals (with endpoints determined by terms in $\overline{c}$), and so a is in one of these. By questioning the complete types of the tuples $\overline{b}, \overline{c}$ we can answer this question in S

Theorem (Dolich, Knight, Lange & Marker)

Given a Scott set S there is a recursively saturated model $\mathcal{M}$ of *Presburger Arithmetic* such that S is the set of subsets of $\mathbb{N}$ which are coded in $\mathcal{M}$.