# End extensions of models

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# Aim in this talk

Survey the contents of:

*Dimitracopoulos C. and V. Paschalis,*
*"End extensions of models of weak arithmetic theories"*

To appear in Notre Dame Journal of Formal Logic.

### Theorem 1 (ACT-Semantic Form)

*Let $M$ be a model of $PA$ and $T$ be a theory definable in $M$. If $M \models Con(T)$, then there exists a model $K$ of $T$ such that $K$ is "**strongly definable**" in $M$.*

$K$ is "strongly definable" in $M$, if:

(a) the universe of $K$ may be taken to be the same as that of $M$ and

(b) the satisfaction relation for $K$ is parametrically definable in $M$.

If the theory $T$ contains $PA$

### Lemma 1

*If $M, K$ are models of $PA$ and $K$ is strongly definable in $M$, then $M$ is isomorphic to an initial segment of $K$.*

*McAloon, K., "Completeness theorems, incompleteness theorems and models of arithmetic", Trans. Amer. Math. Soc. 239 (1978), 253–277.*

**Theorem 2 (McAloon 1978)**

*Let M be a model of PA and T be a theory definable in M s.t.*

$$M \models Con(T + Tr(\Pi_n))$$

*where* $Tr(\Pi_n)$ *denotes the set of (Gödel numbers of)* $\Pi_n$ *sentences true in M.*

*Then there exists a model K of T such that K is (isomorphic to) a proper* $\Sigma_n$*-elementary end extension of M.*

*Paris, J.B.,*
*"Some conservation results for fragments of arithmetic",*
*Model theory and arithmetic (Paris, 1979-1980),*
*251–262, Lecture Notes in Math., 890, Springer,*
*Berlin-New York, 1981.*

**Theorem 3 (Paris 1981)**

*Let M be a model of* $B\Sigma_n$, $n \geq 2$, *and* $T \supseteq I\Delta_0$ *be a theory* $\Delta_{n-1}$ *definable in M such that* $M \models Con(T)$.

*Then there exists a model K of T which is* $\Delta_n$ *definable in M and K is isomorphic to a proper end extension of M.*

*Paris, J.B., and L.A.S. Kirby,*
*“Σn -collection schemas in arithmetic”,*
*Logic Colloquium ’77 (Proc. Conf., Wroclaw, 1977),*
*199–209, North-Holland, Amsterdam-New York, 1978.*

**Theorem 4 (Paris-Kirby 1978)**

*For any countable model* $M$ *of* $I\Delta_0$ *and* $n \geq 2$,

(a) $M \models B\Sigma_n \Leftrightarrow$ *there exists* $K \models I\Delta_0$ *such that* $M \prec_{n,e} K$.

(b) *if* $M$ *has a proper* $\Sigma_1$ *elementary end extension,*
*then* $M \models B\Sigma_2$.

Problem 1

*Is every countable model M of* $B\Sigma_1$ *extendable to a model K of* $I\Delta_0$ *such that* $M \prec_{0,e} K$*?*

Since for any *LA* structures $M$ and $K$, $M \subset_e K$ implies $M \prec_0 K$, it follows that Problem 1 is equivalent to

Problem 2

*Is every countable model M of* $B\Sigma_1$ *extendable to a model K of* $I\Delta_0$ *such that* $M \subset_e K$*?*

*Clote, P., and J. Krajicek,*
*Open problems, Oxford Logic Guides, 23,*
*Arithmetic, proof theory, and computational complexity (Prague, 1991), 1–19, Oxford Univ. Press, NY, 1993.*

*Wilkie, A.J., and J. Paris, "On the existence of end extensions of models of bounded induction", Logic, methodology and philosophy of science, VIII (Moscow, 1987), 143– 161, Stud. Logic Found. Math., 126, North-Holland, Amsterdam, 1989.*

Paris and Wilkie introduced the notion of $\Gamma$-*fullness*, $\Gamma$ being a set of sentences, and proved:

**Theorem 5 (Paris-Wilkie 1989)**

*For any countable non-standard model M of $B\Sigma_1$, if M is $I\Delta_0$-full, then M has a proper end extension $K \models I\Delta_0$.*

**Theorem 6 (Paris-Wilkie 1989)**

*For any countable nonstandard model $M$ of $B\Sigma_1$, if $M \models exp$, then $M$ is $I\Delta_0$-full.*

**Theorem 7 (Paris-Wilkie 1989)**

*Every countable nonstandard model $M$ of $B\Sigma_1 + exp$ has a proper end extension $K \models I\Delta_0$.*

> "**Remark.** *A direct proof that any countable model of $I\Delta_0 + B\Sigma_1$ which is closed under exponentiation has a proper end extension to a model of $I\Delta_0$ may be obtained by mimicking the proof of Theorem 4 but with "Semantic Tableau consistency of $\Gamma$" in place of "$\Gamma$-full" and adding a new constant symbol $\pi > M$ to ensure that the end extension is proper."*

In what follows, $Tabcon(T)$ will denote the $\Pi_1$ sentence $\neg\exists x Tabinconpr(T, x)$ and $LA^* = LA \cup \{c\} \cup \{c_a : a \in M\}$.

$T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq \ldots$ sequence of theories in $LA^*$ such that:

(a) $M \models Tabcon(T_n + \Delta + c{>}M)$, for any $n \in \mathbb{N}$, where $\Delta$ denotes the diagram of $M$ and $c > M$ the set of sentences $\{c{>}c_a : a \in M\}$.

(b) $T_\infty = \bigcup_{n \in \mathbb{N}} T_n$ is a complete theory in $LA^*$, containing the diagram of $M$ and all sentences of the form $c{>}c_a$, $a \in M$,

(c) whenever $\exists y \leq c_a \theta(y, c, \vec{c}) \in T_\infty$, then there exists $b \leq a$ in $M$ such that $\theta(c_b, c, \vec{c}) \in T_\infty$.

One then applies the omitting types theorem to obtain $K^* \models T_\infty$ in which the interpretations of $\{c_a : a \in M\}$ form an initial segment.

$k$-$Tabcon(T)$ denotes the formula expressing "there is no tableau proof of a contradiction from $T$, using only substitution instances of formulas with Gödel number $\leq k$ "

This formula is reminiscent of the formula $Con(X, k)$ used in:

> *Wilkie, A.J., and J.B. Paris, "On the scheme of induction for bounded arithmetic formulas", Ann. Pure Appl. Logic 35 (1987), no. 3, 261–302.*

### Problem 3

*Is every model $M$ of $B\Sigma_1 + exp$ extendable to a model $K$ of $I\Delta_0$ such that $M \subset_e K$?*

A positive solution to Problem 3 would complete the picture offered by:

### Theorem 8

*For every $M \models B\Sigma_n$, $n \geq 2$, there exists $K \models I\Delta_0$ such that $M \prec_{n-1,e} K$.*

### Problem 4

*Does every model of $B\Sigma_n$, $n \geq 2$, have a proper $\Sigma_n$-elementary end extension satisfying $I\Delta_0$?*

*Clote., P., "A note on the MacDowell-Specker theorem, Fund. Math. 127 (1987), no. 2, 163–170.*

*Clote., P., "Addendum to: A note on the MacDowell-Specker theorem", [Fund. Math. 127 (1987), no. 2, 163–170], Fund. Math. 158 (1998), no. 3, 301–302.*

**Theorem 9 (Clote 1998)**

*Every model of $I\Sigma_n$, $n \geq 2$, has a proper $\Sigma_n$-elementary end extension satisfying $I\Delta_0$*

**Theorem 10**
*Every model of $B\Sigma_n$, $n \geq 3$, has a proper $\Sigma_{n-1}$-elementary end extension satisfying $I\Delta_0$.*

If Problem 3 does indeed have a positive solution, this would allow us to deduce:

**Theorem 11**
*For every $n > 0$, if $M \models B\Sigma_n$ ($+exp$, if $n = 1$), then there exists $K \models I\Delta_0$ such that $M \prec_{n-1,e} K$.*

### Problem 5

*Is every countable model $M$ of $B\Sigma_1 + \Omega_1$ extendable to a model $K$ of $I\Delta_0$ such that $M \subset_e K$?*

### Problem 6

*Is every model $M$ of $B\Sigma_1 + \Omega_1$ extendable to a model $K$ of $I\Delta_0$ such that $M \subset_e K$?*

Finally, one could pose variants of these problems for subtheories of $B\Sigma_1 + \Omega_1$, e.g. theories of the form $T^i_2 + B\Sigma^b_i$, where $T^i_2$ denotes the usual theories introduced by S. Buss.