COUNTING IN $\Delta_0^{\sharp \mathbb{N}}$ MODULO MATRICES MONOIDS

Henri-Alex Esbelin

LIMOS, Clermont-Ferrand Universities

Let $\Delta_0^{\mathbb{N}}$ be the class of the relations on the set of the natural numbers $\mathbb{N}$ which are definable by a Δ_0 formula. Its closure under counting is a long standing open problem. E.g. Paris and Wilkie conjectured that the unary relation "x is a prime number of even index" is not Δ_0-definable. Such problems led us to introduce a general notion of counting modulo a finite monoid (see [1], [3], [4] and [5]).
The counting functions defined by $f_R(n;\vec{x}) = card\{i \leq n; R(i;\vec{x})\}$, where R is a relation, are not under the scope of such a notion. Hence we introduced in [2] the class $\Delta_0^{\sharp \mathbb{N}}$ by adding to the expressive power of Δ_0 (mainely bounded quantification) the possibility of counting the positive integers satisfying a given relation.

We prove here the following theorem, which states the closure of $\Delta_0^{\mathbb{N}}$ under counting modulo $SL(2;\mathbb{N})$:

Theorem. Let a, b, c and d be functions with domain $\mathbb{N} \times \mathbb{N}^k$ and codomain $\mathbb{N}$. Suppose that for all i and $\vec{x}$, we have $a(i;\vec{x})d(i;\vec{x}) - b(i;\vec{x})c(i;\vec{x}) = 1$. Let us suppose that the relation $\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} a(i;\vec{x}) & b(i;\vec{x}) \\ c(i;\vec{x}) & d(i;\vec{x}) \end{pmatrix}$ is Δ_0-definable.
Then the relation $\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \prod_{i=0}^{i=y} \begin{pmatrix} a(i;\vec{x}) & b(i;\vec{x}) \\ c(i;\vec{x}) & d(i;\vec{x}) \end{pmatrix}$ is $\Delta_0^{\sharp \mathbb{N}}$-definable.

We then study similar properties for the group $SL(2;\mathbf{Z})$ and the monoid $SL(3;\mathbb{N})$.

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