Counting modulo infinite monoïds and Δ_0-definability

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Plan

- What is Δ_0-definability and why counting modulo (finite) monoïds?

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- A new result about $SL(2,\mathbb{N})$

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- A new result about $SL(2, \mathbb{N})$
- Counting modulo infinite monoïds
- A new result about $SL(2, \mathbb{Z})$
- Conclusion and further work

What is Δ_0−definability?

x is not prime nor 0 nor 1

$$(\exists u)_{u<x}\,(\exists v)_{v<x}\,(x = uv)$$

What is Δ_0−definability?

x is not prime nor 0 nor 1

$$(\exists u)_{u<x}(\exists v)_{v<x}(x = uv)$$

Δ_0−definability is essentially definability with a formula in the langage of arithmetic where the quantified variables are bounded by terms

What is Δ_0−definability?

Major open problem

Find a "simple" artithmetical relation

NOT Δ_0−definable

Here "simple" is a non defined meta-mathematical notion!

What is Δ_0−definability?

Open exemple

The relation *y is the n-th prime number*

IS NOT KNOWN TO BE Δ_0−definable

What is Δ_0-definability?

Open exemple

The relation y *is a prime number of even index*

IS NOT KNOWN TO BE Δ_0-definable

Why counting modulo (finite) monoïds?

The relation y *is a prime number of even index*

is defined by $(y \textit{ is prime}) \wedge (f(y) = 0)$ where

f is recursively defined by

$$\begin{cases} f(0) = 0 \\ \\ f(i+1) = f(i) + 1 \bmod 2 \text{ if } i \text{ is prime} \\ f(i+1) = f(i) \text{ if } i \text{ is not prime} \end{cases}$$

Why counting modulo (finite) monoïds?

The relation y *is a prime number of even index*

is defined by

$$(y \textit{ is prime}) \wedge (g(0) +_2 g(1) +_2 \ldots +_2 g(y) = 0)$$

$$\text{where } \begin{cases} g(i) = 1 \in \mathbb{Z}/2\mathbb{Z} \text{ if } i \text{ is prime} \\ g(i) = 0 \in \mathbb{Z}/2\mathbb{Z} \text{ if } i \text{ is not prime} \end{cases}$$

$+_2$ is the sum modulo 2.

Why counting modulo (finite) monoïds?

$\Delta_0^{\sharp G}$ is defined by adding to the definition of Δ_0

the following closure under counting modulo G

where G is a finite monoïd:

Why counting modulo (finite) monoïds?

$$\text{Suppose that } \left\{ \begin{array}{ccc} \mathbb{N} & \to & G \\ i & \to & g(i) \end{array} \right. \text{ is s.t.}$$

for all $g \in G$, the relation $g(i) = g$ is $\Delta_0^{\sharp G}$−definable

Why counting modulo (finite) monoïds?

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for all $g \in G$, the relation $g(i) = g$ is $\Delta_0^{\sharp G}$−definable

THEN

for all $g \in G$, the relation $g(0) +_G g(1) +_G \ldots +_G g(y) = g$ is $\Delta_0^{\sharp G}$−definable

Counting modulo $\mathbb{N}$

Strong counting modulo $\mathbb{N}$

Counting modulo $\mathbb{N}$

Strong counting modulo $\mathbb{N}$

Suppose that $\left\{ \begin{array}{ccc} \mathbb{N} & \rightarrow & \mathbb{N} \\ i & \rightarrow & g(i) \end{array} \right.$ is s.t.

the relation $z = g(i)$ is $\Delta_0^{\sharp\mathbb{N}}$−definable

Counting modulo $\mathbb{N}$

Strong counting modulo $\mathbb{N}$

Suppose that $\left\{ \begin{array}{ccc} \mathbb{N} & \to & \mathbb{N} \\ i & \to & g(i) \end{array} \right.$ is s.t.

the relation $z = g(i)$ is $\Delta_0^{\sharp\mathbb{N}}$−definable

THEN

the relation $g(0) + g(1) + \ldots + g(y) = z$

is $\Delta_0^{\sharp\mathbb{N}}$−definable

Counting modulo $\mathbb{N}$

Weak counting modulo $\mathbb{N}$

Counting modulo $\mathbb{N}$

Weak counting modulo $\mathbb{N}$

Suppose that $\left\{ \begin{array}{ccc} \mathbb{N} & \rightarrow & \{0;1\} \\ i & \rightarrow & g(i) \end{array} \right.$ is s.t.

the relation $1 = g(i)$ is $\Delta_0^{\sharp\mathbb{N}}$−definable

Counting modulo $\mathbb{N}$

Weak counting modulo $\mathbb{N}$

Suppose that $\left\{ \begin{array}{ccc} \mathbb{N} & \rightarrow & \{0;1\} \\ i & \rightarrow & g(i) \end{array} \right.$ is s.t.

the relation $1 = g(i)$ is $\Delta_0^{\sharp\mathbb{N}}-$definable

THEN

the relation $g(0) + g(1) + ... + g(y) = z$

is $\Delta_0^{\sharp\mathbb{N}}-$definable

Counting

Known facts

Counting

Known facts

JAF's folklaw:

Previous definitions of counting modulo $\mathbb{N}$ are equivalent

Counting

Known facts

JAF's folklaw+ **Theorem** (Clote 95)

$$\Delta_0^{\sharp\mathbb{Z}/2\mathbb{Z}} \subseteq \Delta_0^{\sharp\mathbb{Z}} \subseteq \Delta_0^{\sharp\sigma_5}$$

σ_5 is the permutation group over five elements.

$$\begin{matrix} \Delta_0^{\sharp\mathbb{Z}/2\mathbb{Z}} & & \\ & \subseteq & \Delta_0^{\sharp\mathbb{Z}/6\mathbb{Z}} \\ \Delta_0^{\sharp\mathbb{Z}/3\mathbb{Z}} & & \end{matrix}$$

A result about $SL(2,\mathbb{N})$

Theorem

$$\text{Let } \begin{cases} \mathbb{N} & \rightarrow & SL(2,\mathbb{N}) \\ i & \rightarrow & \begin{pmatrix} a(i) & b(i) \\ c(i) & d(i) \end{pmatrix} \end{cases}$$

$$\text{where } \begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} a(i) & b(i) \\ c(i) & d(i) \end{pmatrix} \text{ is } \Delta_0\text{–definable}$$

A result about $SL(2,\mathbb{N})$

Theorem

Then

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} a(0) & b(0) \\ c(0) & d(0) \end{pmatrix} \begin{pmatrix} a(1) & b(1) \\ c(1) & d(1) \end{pmatrix} \cdots \begin{pmatrix} a(y) & b(y) \\ c(y) & d(y) \end{pmatrix}$$

is $\Delta_0^{\sharp\mathbb{N}}$—definable

A result about $SL(2, \mathbb{N})$

Idea of proof

A result about $SL(2,\mathbb{N})$

Idea of proof

Lemma 1

$$\text{Let } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{N})$$

There exist $k(a,b,c,d)$ and $\alpha(i,a,b,c,d)$ s.t.

$$M = \begin{pmatrix} 1 & \alpha(0,a,b,c,d) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha(1,a,\dots) & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & \alpha(2k(a,\dots),a,\dots) \\ 0 & 1 \end{pmatrix}$$

and $\alpha(0) \geq 0$, $\alpha(1) \geq 1$, ... $\alpha(2k-1) \geq 1$, $\alpha(2k) \geq 0$,

and α and k have Δ_0−definable graph.

A result about $SL(2,\mathbb{N})$

Idea of proof

Lemma 2

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{N})$, and α a function with a Δ_0-definable graph

and $\alpha(0) \geq 0$, $\alpha(1) \geq 1$, ... $\alpha(2k-1) \geq 1$, $\alpha(2k) \geq 0$,

$$\text{Then } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \alpha(0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha(1) & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \alpha(2k) \\ 0 & 1 \end{pmatrix}$$

is Δ_0-definable

A result about $SL(2,\mathbb{N})$

Idea of proof

Lemma 2bis

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{N})$, and α a function with a Δ_0-definable graph

and $\alpha(0) \geq 0$, $\alpha(1) \geq 0$, ... $\alpha(2k-1) \geq 0$, $\alpha(2k) \geq 0$,

$$\text{Then } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \alpha(0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha(1) & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \alpha(2k) \\ 0 & 1 \end{pmatrix}$$

is $\Delta_0^{\sharp\mathbb{N}}-$definable

What could be counting modulo infinite monoïds?

What could be counting modulo infinite matrix monoïds?

What could be counting modulo infinite monoïds?

What could be counting modulo infinite matrix monoïds?

$$\text{Let } M(i) = (a_{u,v}(i))_{1 \leq u,v \leq d} \in G$$

$$\text{Suppose } (z_{u,v})_{1 \leq u,v \leq d} = (a_{u,v}(i))_{1 \leq u,v \leq d} \text{ is } \Delta_0^{\sharp G}\text{−definable}$$

What could be counting modulo infinite monoïds?

What could be counting modulo infinite matrix monoïds?

$$\text{Let } M(i) = (a_{u,v}(i))_{1 \leq u,v \leq d} \in G$$

$$\text{Suppose } (z_{u,v})_{1 \leq u,v \leq d} = (a_{u,v}(i))_{1 \leq u,v \leq d} \text{ is } \Delta_0^{\sharp G}\text{−definable}$$

THEN

$$(z_{u,v})_{1 \leq u,v \leq d} = M(0)M(1)...M(y) \text{ is } \Delta_0^{\sharp G}\text{−definable}$$

What could be counting modulo infinite monoïds?

What could be counting modulo infinite finitely generated monoïds?

Let G be a monoïd with a finite set of generators $\Gamma = \{\gamma_1, \dots, \gamma_d\}$

What could be counting modulo infinite monoïds?

What could be counting modulo infinite finitely generated monoïds?

Let G be a monoïd with a finite set of generators $\Gamma = \{\gamma_1, ..., \gamma_d\}$

$$\text{For all } \begin{cases} N & \rightarrow & \Gamma \\ i & \rightarrow & g(i) \end{cases} \text{ s.t.}$$

for all $\gamma \in \Gamma$, the relation $g(i) = \gamma$ is $\Delta_0^{\sharp G}$−definable

THEN

for all $\gamma \in \Gamma$, the relation $g(0).g(1)....g(z) = \gamma$ is $\Delta_0^{\sharp G}$−definable

What could be counting modulo infinite monoïds?

What could be counting modulo infinite finitely generated monoïds of matrices?

The first definition is stronger than the second

A result about $SL(2, \mathbb{Z})$

Definition

What is Δ_0-definability in $\mathbb{Z}$?

A result about $SL(2, \mathbb{Z})$

Definition

What is Δ_0–definability in $\mathbb{Z}$?

just consider a relative integer as couple in $\{0; 1\} \times \mathbb{N}$

A result about $SL(2, \mathbb{Z})$

Definition

$$\text{Hence, } a : \mathbb{N} \mapsto \mathbb{Z} \text{ has a } \Delta_0\text{—definable graph}$$

$$\text{iff } |a| \text{ has a } \Delta_0\text{—definable graph}$$
$$\text{and } sign(a(i)) = 1 \text{ is a } \Delta_0\text{—definable relation}$$

A result about $SL(2, \mathbb{Z})$

Theorem

A result about $SL(2,\mathbb{Z})$

Theorem

$$\text{Let } \begin{cases} \mathbb{N} & \rightarrow & SL(2,\mathbb{Z}) \\ i & \rightarrow & \begin{pmatrix} a(i) & b(i) \\ c(i) & d(i) \end{pmatrix} \end{cases}$$

where a, b, c, d are polynomialy bounded,

and $\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} a(i) & b(i) \\ c(i) & d(i) \end{pmatrix}$ is a Δ_0-definable relation

A result about $SL(2,\mathbb{Z})$

Theorem

THEN

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} a(0) & b(0) \\ c(0) & d(0) \end{pmatrix} \begin{pmatrix} a(1) & b(1) \\ c(1) & d(1) \end{pmatrix} \cdots \begin{pmatrix} a(y) & b(y) \\ c(y) & d(y) \end{pmatrix}$$

is $\Delta_0^{\sharp\mathbb{N}}$definable

A result about $SL(2,\mathbb{Z})$

Idea of proof

Generalize the results about Δ_0−definability concerning

the standard euclidean algorithm

to the least absolute remainder euclidean algorithm:

A result about $SL(2,\mathbb{Z})$

Idea of proof

$$r_n = \beta_n r_{n+1} + r_{n+2}$$

where $\frac{|r_{n+1}|}{2} < |r_{n+2}| \leq \frac{|r_{n+1}|}{2}$

Conclusion and further work

Conclusion and further work

The proof does not extend to

$$SL(3, \mathbb{Z})$$

Conclusion and further work

Question 1: Find a convenient definition for

Multiple Continued Fractions

for solving the $SL(3,\mathbb{Z})$ case.

Conclusion and further work

Question 2: Compare both definitions of counting

modulo infinite monoïds

in the case of finitely generated

monoïds of matrices

Conclusion and further work

Work: Consider counting modulo infinite monoïds

in the case of a finite presentation

with generators and relations