(Im)predicativity and Fregean arithmetic

Fernando Ferreira

Universidade de Lisboa

Journées sur les Arithmétiques Faibles 33
University of Gothenburg

June 16-18, 2014

Frege's alertness

"As far as I can see, a dispute can arise only concerning my basic law of value-ranges (V), which perhaps has not yet been explicitly formulated by logicians although one thinks in accordance with it if, e.g., one speaks of extensions of concepts. I take it to be purely logical. At any rate, the place is hereby marked where there has to be a decision."

Gottlob Frege in the forward of volume I of *Grundgesetze der Arithmetik*.

Frege’s lamentation

”Even now, I do not see how arithmetic can be formulated scientifically, how the numbers can be apprehended as logical objects and brought under consideration, if it is not – at least conditionally – permissible to pass from concept to its extension. May I always speak of the extension of a concept, of a class? And if not, how are the exceptions to be recognised?”

Gottlob Frege in the afterword of volume II of *Grundgesetze der Arithmetik*

Frege's logical system

- Second-order *impredicative* logic:

$$\exists F \forall x (Fx \leftrightarrow \theta(x)),$$

  where θ is any formula in which 'F' does not occur free.

- An *extension operator* governed by Basic Law V:

$$\hat{x}.\theta(x) = \hat{x}.\psi(x) \leftrightarrow \forall x(\theta(x) \leftrightarrow \psi(x)),$$

  where θ and ψ are any formulas of the language.

Predicative comprehension amounts to restricting the comprehension scheme above to formulas θ in which second-order bound variables do not occur (not even in extension terms).

The logicist *membership* relation

$$x \in y \;:\equiv\; \exists F(y = \hat{w}.Fw \wedge Fx)$$

It conveys a priority of concepts over sets.

The *principle of concretion*:

$$x \in \hat{w}.\theta(w) \;\leftrightarrow\; \theta(x)$$

$\leftarrow)$ We need Basic Law V and the following comprehension:

$$\exists F \forall w(Fw \leftrightarrow \theta(w))$$

Russell's paradox

$$\rho(x) :\equiv x \notin x$$

$$r := \hat{x}.\rho(x)$$

$$\forall w\,(w \in r \leftrightarrow \rho(w))$$

$$\forall w\,(w \in r \leftrightarrow w \notin w)$$

$$r \in r \leftrightarrow r \notin r$$

Note that $\rho(x)$ is $\forall G(x = \hat{w}.Gw \rightarrow \neg Gx)$.

Avoiding the paradox

- Abandon the extension operator (Russell, Whitehead, Ramsey). Neologicism.
- Restrict to predicative comprehension (Dummett, Heck).
- Maintain impredicative comprehension but restrict the extension operator to predicative formulas.

On Heck's predicative system

Theorem (Heck, 96)

Heck's predicative system is consistent.

Theorem (Heck, 96)

Robinson's theory Q *is interpretable in Heck's predicative system.*

A nontrivial amount of arithmetic and analysis is interpretable in Q. This is due to work of Nelson, Solovay, Wilkie, Pudlák, Visser, Ferreira, etc. For a survey see [FF & GFerreira, 2013].

Theorem (FF, to appear in NDJFL)

Heck's simple predicative system is not able to interpret $\mathrm{I}\Delta_0(\mathrm{super}^2\mathrm{exp})$.

An intermediate theory

(a) $\langle x, y\rangle = \langle u, v\rangle \to x = u \wedge y = v$.

(b) predicative comprehension.

(c) *modified* Σ^1_1-choice:

$$\forall x \exists F \phi(F, x) \to \exists R \forall x \exists y\, \phi(R_{x,y}, x),$$

for formulae ϕ without second-order quantifiers. Here $R_{x,y}(u)$ abbreviates $R(\langle u, \langle x, y\rangle\rangle)$.

(d) $\ddagger F = \ddagger G \leftrightarrow \forall x(Fx \leftrightarrow Gx)$.

(e) for each natural number k, $\exists_k x \forall F(x \neq \ddagger F)$.

- It is possible to interpret Heck's predicative theory, *without* impredicative value-range terms, in the above theory.
- One can introduce the impredicative value-range terms by the finitistic methods of John Burgess, using the so-called *splitting lemma* and *injection lemma*.

Shoenfield's theorem

Given T a first-order theory, we define T^P the theory obtained from T by extending the language to (polyadic) second-order language, adding the predicative comprehension principle and replacing any (unrestricted) schemes of the original theory T by the corresponding single axioms.

If T is PA then T^P is ACA_0.

If T is ZF then T^P is BG.

Theorem
T^P *is conservative over* T.

Modified Σ^1_1-choice. Model theory.

Modified Σ^1_1-choice:

$$\forall x \exists F \phi(F, x) \to \exists R \forall x \exists y \phi(R_{x,y}, x),$$

for formulae ϕ without second-order quantifiers.

Lemma

Models of T^{P} *satisfying modified* Σ^1_1*-choice also satisfy* Δ^1_1*-comprehension.*

Theorem

If $\mathcal{M}$ *is a* recursively saturated structure*, then Def*$(\mathcal{M})$ *is a model of modified* Σ^1_1*-choice (and of predicative comprehension).*

Modified Σ^1_1-choice. Proof Theory.

This is a very rough sketch.

Add to the predicative sequent calculus the following rule:

$$\frac{\Gamma \Rightarrow \Delta, \exists F \phi(F, a)}{\Gamma \Rightarrow \Delta, \exists R \forall x \exists y \phi(R_{x,y}, x)}$$

where ϕ is without second-order variables and a is an eigenvariable. This extended calculus proves modified Σ^1_1-choice.

By a *partial* cut-elimination theorem, if the conclusion is of the form $\exists F \theta(F)$, with θ without second-order quantifiers, then there is a proof in the extended calculus where each formula of the sequents has that form.

In this situation, one can show that the new rule is superfluous (in the presence of pairing).

An Herbrand like theorem

Theorem
Suppose that a formula $\exists F \phi(F)$, where ϕ does not have second-order quantifiers, is provable in the sequent calculus of pure predicative logic. Then there are abstracts $\{x : \theta_1(x, \overline{z}_1)\}, \ldots, \{x : \theta_n(x, \overline{z}_n)\}$ such that the sequent

$$\Rightarrow \exists \overline{z}_1 \phi(\{x : \theta_1(x, \overline{z}_1)\}), \ldots, \exists \overline{z}_n \phi(\{x : \theta_n(x, \overline{z}_n)\})$$

is provable in the restriction *of the above calculus to the language without second-order quantifiers.*

(Eq) $\forall F \forall x \forall y (x = y \wedge Fx \rightarrow Fy)$

(LV) $\forall F \forall G (\hat{x}.Fx = \hat{x}.Gx \leftrightarrow \forall x (Fx \leftrightarrow Gx))$

Heck's ramified system

Ramified comprehension:

$$\exists F^n \forall x (F^n x \leftrightarrow \theta(x)),$$

where θ is a formula in which does not occur second-order variables of level $> n$, nor bound variables of level $\geq n$.

Theorem (Heck, 96)

Heck's ramified predicative theory is consistent.

Theorem (after Burgess & Hazen, 98)

Heck's ramified predicative arithmetic interprets $\mathrm{I}\Delta_0(\mathrm{exp})$.

Theorem (FF, to appear in NDJFL)

Heck's ramified predicative system is not able to interpret $\mathrm{I}\Delta_0(\mathrm{super}^3\mathrm{exp})$.

On reducibilities

The axiom (scheme) of reducibility:

$$\forall F^n \exists G^0 \forall x (G^0 x \leftrightarrow F^n x).$$

Finite reducibility: If $\theta(x)$ is true of only finitely many elements, then $\theta(x)$ is co-extensive with a predicative concept of the form:

$$x = a_1 \vee x = a_2 \vee \ldots \vee x = a_n$$

For every formula $\theta(x)$,

$$\mathrm{Finite}_x(\theta) \rightarrow \exists F \forall x (Fx \leftrightarrow \theta(x))$$

Finite reducibility: a false start

To postulate that concepts of the form "there is only finitey many w such that $\theta(x, w)$" are co-extensive with predicative concepts when θ is predicative.

Essentially in [FF, 2005]. Goes beyond finite reducibility.

John Burgess in [Burgess, 2005]: "For it seems clear to me that no idea could be more non-, un-, and anti-Fregean than that of helping oneself to intuitions about finitude as axioms, not proved as theorems from logical axioms and a suitable definition of finitude."

A mixed system

- Two types of concept variables:
  1. variables which range over **impredicative concepts**: $\mathfrak{F}$, $\mathfrak{G}$, etc. The corresponding impredicative comprehension scheme is:

     $$\exists \mathfrak{F} \forall x (\mathfrak{F} x \leftrightarrow \theta(x)),$$

     where $\theta(x)$ is any formula not containing '$\mathfrak{F}$' free.
  2. variables which range over **predicative concepts**: F, G, etc. The corresponding predicative comprehension scheme is:

     $$\exists F \forall x (F x \leftrightarrow \theta(x)),$$

     where $\theta(x)$ is a formula in the predicative fragment of the language which contains neither free 'F' nor bound second-order quantifiers.

No contradiction is forthcoming

- **The extension operator is restricted to formulas of the predicative fragment only**. I.e., one can form $\hat{x}.\theta(x)$ only if no impredicative variables occur in θ.

Theorem (FF)

The mixed system PE *is consistent.*

Proof.

The mixed system is obtained by extending Heck's simple predicative system with second-order impredicative variables and the corresponding comprehension principle. Given a model of Heck's predicative system, interpret the impredicative variables as ranging over the power set of the first-order domain of the model. □

The development of Fregean arithmetic (I)

$$Nx.Fx := \hat{z}.\exists H(H\ \mathbf{eq}\ F \wedge z = \hat{w}.Hw)$$

Theorem (ess. Heck, 96)

$$\mathrm{PE} \vdash \forall F \forall G(Nx.Fx = Nx.Gx \ \leftrightarrow\ F\ \mathbf{eq}\ G)$$

$$0 := Nx.(x \neq x)$$

$$S(x, y) \ :\equiv\ \exists F(y = Nw.Fw \wedge \exists z(Fz \wedge x = Nw.(Fw \wedge w \neq z)))$$

The natural number concept

$$\mathbf{Her}(\mathfrak{F}) :\equiv \forall x \forall y (\mathfrak{F}x \land S(x, y) \rightarrow \mathfrak{F}y)$$

$$\mathbb{N}(x) :\equiv \forall \mathfrak{F}(\mathfrak{F}0 \land \mathbf{Her}(\mathfrak{F}) \rightarrow \mathfrak{F}x)$$

$$x \leq y :\equiv \forall \mathfrak{F}(\mathfrak{F}x \land \mathbf{Her}(\mathfrak{F}) \rightarrow \mathfrak{F}y).$$

Theorem (Finite reducibility)

$$\mathrm{PE} \vdash \forall y (\mathbb{N}(y) \rightarrow \exists F \forall x (Fx \leftrightarrow x \leq y)).$$

Proof.
By induction on y. □

The development of Fregean arithmetic (II)

Corollary

PE $\vdash \forall x(\mathbb{N}(x) \to \exists y S(x, y))$.

Proof.

(Sketch) Given a natural number x, we use the Fregean trick of taking y as $Nu.(u \leq x)$. Note that this makes sense, thanks to finite reducibility. □

Sets, reducibility and stipulation

Theorem

$$\mathrm{PE} \vdash \exists z \forall x(x \in z \leftrightarrow \theta(x)) \leftrightarrow \exists F \forall x(Fx \leftrightarrow \theta(x)),$$

for any formula θ in which 'z' and 'F' do not occur free.

Beyond logicism:

Reducibility/set existence stipulation: $\exists F \forall x(Fx \leftrightarrow \theta(x))$.

Can we consistently adjoin $\exists F \forall x(Fx \leftrightarrow \mathbb{N}(x))$?

References

John Burgess: *Fixing Frege*. Princeton University Press, 2005.

John Burgess & Allen Hazen: “Predicative logic and formal arithmetic.” *Notre Dame Journal of Formal Logic* 39: 1-17 (1998).

Luís Cruz-Filipe & Fernando Ferreira: “The finitistic consistency of Heck’s predicative Fregean system”. To appear in *Notre Dame Journal of Formal Logic*.

Fernando Ferreira: “Amending Frege’s *Grundgesetze der Arithmetik*.” *Synthese* 147:3-19 (2005).

References (continuation)

Fernando Ferreira & Gilda Ferreira: “Interpretability in Robinson’s Q.” *The Bulletin of Symbolic Logic* 19: 289-317 (2013).

Gottlob Frege: *Basic Laws of Arithmetic*. Edited and translated by Philip A. Ebert and Marcus Rossberg(13). With a forward by Crispin Wright. Oxford University Press, 2013.

Richard Heck: “On the consistency of predicative fragments of Frege’s *Grundgesetze der Arithmetik*.” *History and Philosophy of Logic* 17:209-220 (1996).

THANK YOU