Weak Analysis: Metamathematics

Fernando Ferreira

Universidade de Lisboa

Journées sur les Arithmétiques Faibles 33
University of Gothenburg

June 16-18, 2014

Second-order arithmetic

- Z_2: Hilbert-Bernays
- *The big five*:

  Π^1_1-CA_0
  ATR_0
  ACA_0
  WKL_0
  RCA_0

- $RCA_0^\star$
- *Weak analysis*:

  BTPSA
  TCA^2
  BTFA

Weak Analysis

"To find a mathematically significant subsystem of analysis whose class of provably recursive functions consist only of the computationally feasible ones."

Wilfried Sieg (1988)

- BTFA: Base theory for feasible analysis

  Real numbers, continuous functions, intermediate-value theorem. With (versions of) weak König's lemma: Heine-Borel theorem, uniform continuity theorem.

- TCA^2: Theory of counting arithmetic (analysis)

  Riemann integration and the fundamental theorem of calculus.

- BTPSA: Base theory for polyspace analysis

Basic set-up (fourteen open axioms)

$$
\begin{array}{rcl}
x\varepsilon = x & & x \times \varepsilon = \varepsilon \\
x(y0) = (xy)0 & & x \times y0 = (x \times y)x \\
x(y1) = (xy)1 & & x \times y1 = (x \times y)x \\
x0 = y0 \to x = y & & x1 = y1 \to x = y \\
x \subseteq \varepsilon & \leftrightarrow & x = \varepsilon \\
x \subseteq y0 & \leftrightarrow & x \subseteq y \vee x = y0 \\
x \subseteq y1 & \leftrightarrow & x \subseteq y \vee x = y1 \\
x0 & \neq & y1 \\
x0 & \neq & \varepsilon \\
x1 & \neq & \varepsilon
\end{array}
$$

We abbreviate $x \leq y$ for $1 \times x \subseteq 1 \times y$. We write $x \equiv y$ for $x \leq y \wedge y \leq x$.

Basic set-up (induction on notation)

We abbreviate $x \subseteq^* y$ for $\exists w(wx \subseteq y)$. A *subword quantification* is a quantification of the form $\forall x \subseteq^* t(\ldots)$ or $\exists x \subseteq^* t(\ldots)$.

Definition

A Σ_1^b-formula is a formula of the form $\exists x \leq t\, \phi(x)$, where ϕ is a subword quantification (sw.q.) formula.

Note

Σ_1^b-formulas define the NP-sets.

Definition

The theory Σ_1^b-NIA is the theory constituted by the basic fourteen axioms and the following form of induction on notation:

$$\phi(\varepsilon) \wedge \forall x(\phi(x) \to \phi(x0) \wedge \phi(x1)) \to \forall x \phi(x),$$

where $\phi \in \Sigma_1^b$.

The polytime functions

- Initial functions

  $C_0(x) = x0$ and $C_1(x) = x1$

  Projections

  $Q(x, y) = 1 \leftrightarrow x \subseteq y$; $Q(x, y) = 0 \vee Q(x, y) = 1$
- Derived functions

  By composition

  By bounded recursion on notation:

  $f(\bar{x}, \epsilon) = g(\bar{x})$

  $f(\bar{x}, y0) = h_0(\bar{x}, y, f(\bar{x}, y))_{|t(\bar{x},y)}$

  $f(\bar{x}, y1) = h_1(\bar{x}, y, f(\bar{x}, y))_{|t(\bar{x},y)}$,

  where t is a term of the language and $q|_t$ is the truncation of q at the length of t.

Note

We can introduce, via an extension by definitions, the polytime functions in the theory Σ_1^b-NIA*. Actually, we can see the latter theory as the extension of a quantifier-free calculus* PTCA*.*

U LISBOA | UNIVERSIDADE DE LISBOA

Buss' witness theorem

Theorem

If Σ_1^b-NIA$\vdash \forall x \exists y \theta(x,y)$, *where* $\theta \in \Sigma_1^b$, *then there is a polytime description* f *such that* PTCA $\vdash \theta(x, f(x))$,

Proof.

If there is a proof of $\exists y \theta(x,y)$ in Σ_1^b-NIA, then there is a proof of the sequent $\Rightarrow \exists y \theta(x,y)$ in a suitable calculus with the induction rule:

$$\frac{\Gamma, \phi(v) \Rightarrow \Delta, \phi(v0) \qquad \Gamma, \phi(v) \Rightarrow \Delta, \phi(v1)}{\Gamma, \phi(\epsilon) \Rightarrow \Delta, \phi(s)}$$

where $\phi \in \Sigma_1^b$ and v is an eigenvariable.

By a partial cut-elimination theorem, we obtain a proof whose sequents have $\exists\Sigma_1^b$-formulas only. We can carry along the proof a polytime witness for these sequents. □

Bounded formulas

Definition

A *bounded formula* is a formula obtained from the atomic formulas using propositional connectives and bounded quantifications, i.e., quantifications of the form $\exists x \leq t\, \phi(x)$ or $\forall x \leq t\, \phi(x)$.

These formulas define the predicates in the polytime hierarchy.

Definition

The *bounded collection scheme* $\mathrm{B}\Sigma_1$ is constituted by the formulas:

$$\forall x \leq a \exists y \rho(x, y) \rightarrow \exists b \forall x \leq a \exists y \leq b\, \rho(x, y),$$

where ρ is a bounded formula.

- A Σ_1-formula is a formula of the form $\exists x \rho(x)$, where ρ is a bounded formula. These formulas define the recursively enumerable sets. Π_1- formulas are defined dually.
- A Π_2-formula is a formula of the form $\forall x \exists y \rho(x, y)$, where ρ is a bounded formula.

Buss' theorem on bounded collection

Theorem

Σ_1^b-NIA $+$ $B\Sigma_1$ *is* Π_2*-conservative over* Σ_1^b-NIA.

Proof.

Suppose that Σ_1^b-NIA $+$ $B\Sigma_1 \vdash \forall x \exists y \rho(x, y)$, where ρ is a bounded formula. Then there is a proof of the sequent $\Rightarrow \exists y \rho(x, y)$ in a suitable calculus with the collection rule:

$$\frac{\Gamma, v \leq a \Rightarrow \Delta, \exists y \rho(v, y)}{\Gamma \Rightarrow \Delta, \exists b \forall x \leq a \exists y \leq b\, \rho(x, y)}$$

where ρ is a bounded formula and v is an eigenvariable.

By a partial cut-elimination argument, we obtain a proof whose sequents are constituted by Σ_1-formulas only. We can carry along this proof a suitable bound. E.g., if the sequent $\exists x \theta(x) \Rightarrow \exists y \phi(y)$ appears in the proof, then there is a term t such that

$$\Sigma_1^b\text{-NIA} \vdash \forall c \forall x \leq c(\theta(x) \rightarrow \exists y \leq t(c)\phi(y)).$$

The second-order theory BTFA

Definition
BTFA is the second-order theory whose axioms are Σ_1^b-NIA $+$ BΣ_1 (allowing second-order parameters) plus the following recursive comprehension scheme:

$$\forall x\, (\exists y \phi(x, y) \leftrightarrow \forall z \varphi(x, z)) \rightarrow \exists X \forall x\, (x \in X \leftrightarrow \exists y \phi(x, y))$$

where ϕ is a $\exists\Sigma_1^b$-formula and φ is a $\forall\Pi_1^b$-formula, possibly with first and second-order parameters, and X does not occur in ϕ or φ.

Theorem
The theory BTFA *is first-order conservative over* Σ_1^b-NIA $+$ BΣ_1*.*

The second-order theory BTFA (continued)

Proof.
Given $\mathcal{M}$ a model of Σ^b_1-NIA $+$ BΣ_1, consider $\mathcal{S}$ the set of subsets of the domain of $\mathcal{M}$ which can be defined simultaneously by an $\exists\Sigma^b_1$-formula and a $\forall\Pi^b_1$-formula (with parameters). The structure $\langle \mathcal{M}, \mathcal{S} \rangle$ is a model of BTFA.

We show that, for each sw.q.-formula ϕ with second-order parameters, there are equivalent formulas ϕ_Σ and ϕ_Π ($\exists\Sigma^b_1$ and $\forall\Pi^b_1$, resp.) without second-order parameters. This uses bounded collection to deal with the closure under subword quantification.

Bounded collection is also needed to verify induction on notation. Suppose one has $\phi(\varepsilon) \wedge \neg\phi(x)$, with ϕ a Σ^b_1-formula. Then $\phi(x)$ is of the form $\exists y \leq t(x)\varphi(x, y)$, with φ a sw.q.-formula. We get in $\mathcal{M}$

$$\forall w \subseteq x \forall y \leq t(w)[\varphi_\Sigma(w, y) \leftrightarrow \varphi_\Pi(w, y)].$$

Now argue, using bounded collection, that we can bound the unbounded existential quantifier in $\varphi_\Sigma(w, y)$, for w and y ranging as above.

Weak König's lemma

Given a formula $\phi(x)$, $\mathit{Tree}(\phi_x)$ abbreviates:

$$\forall x \forall y\, (\phi(x) \wedge y \subseteq x \rightarrow \phi(y)) \wedge \forall b \exists x \equiv b\, \phi(x).$$

$\mathit{Path}(X)$ abbreviates:

$$\mathit{Tree}((x \in X)_x) \wedge \forall x \forall y\, (x \in X \wedge y \in X \rightarrow x \subseteq y \vee y \subseteq x).$$

Definition

Weak König's lemma for trees defined by bounded formulas, denoted by Σ_0-WKL, is the following scheme:

$$\mathit{Tree}(\phi_x) \rightarrow \exists X\, (\mathit{Path}(X) \wedge \forall x\, (x \in X \rightarrow \phi(x))),$$

where ϕ is a bounded formula and X is a new second-order variable.

Another conservation result

Theorem
The theory BTFA $+\Sigma_0$-WKL *is first-order conservative over* BTFA.

Proof.
Given $\langle \mathcal{M}, \mathcal{S} \rangle$ a countable model of BTFA and given T a subset of the first-order domain which is an (infinite) tree defined by a bounded formula, it is possible to obtain a subset G of the first-order domain such that G is a infinite path of T and

$$\langle \mathcal{M}, \mathcal{S} \cup \{G\} \rangle \vdash \Sigma_1^b\text{-NIA} + \mathrm{B}\Sigma_1.$$

(By Harrington forcing.)

One can close this structure to get a model $\langle \mathcal{M}, \mathcal{S}^\star \rangle$ of BTFA.

One can iterate this construction ω^2 times to get a model of BTFA $+\Sigma_0$-WKL. □

Harrington forcing

- Fix a countable model $\langle \mathcal{M}, \mathcal{S} \rangle$ of BTFA.
- The *forcing conditions* are given by the set $\mathbb{T}$ of infinite trees defined by bounded formulas. A condition Q is stronger than a condition T if Q is contained in T.
- The *generic filter* is taken with respect to definable dense sets, where the notion of definable is sufficiently general to be closed under quantifications over the conditions (mere second-order definability is not enough).
- The *forcing language* includes constants for the elements of the domain of $\mathcal{M}$ and of $\mathcal{S}$, and an extra second-order constant C (for the generic set).
- $T \Vdash x \in C$ is $\exists b \forall w \equiv b \, (w \in T \rightarrow x \subseteq w)$.

Harrington forcing (continued)

- If $\mathbb{G}$ is a generic filter, then $G := \bigcap \mathbb{G}$ is an infinite path. This uses the fact that, for each b in the domain of $\mathcal{M}$,

$$\mathbb{D}_b := \{T \in \mathbb{T} : (\mathcal{M}, \mathcal{S}) \models \exists^1 x\,(x \equiv b \wedge x \in T)\}.$$

  is dense. Bounded collection is used to show this.

- That $\langle \mathcal{M}, \mathcal{S} \cup \{G\} \rangle$ satisfies Σ^b_1-NIA is obvious (no forcing is needed).

- This is a *weak forcing notion*, i.e.,

$$T \Vdash \phi \quad \text{if, and only if,} \quad \forall Q \leqslant T\, \exists R \leqslant Q\,(R \Vdash \phi).$$

- $T \Vdash \phi$, for ϕ a Σ_1-formula, is a Σ_1-formula. From the *proof* of this fact, it can easily be argued that the structure $\langle \mathcal{M}, \mathcal{S} \cup \{G\} \rangle$ satisfies bounded collection.

Counting and polyspace computability

The *classe of polyspace computable functions* is obtained by adding to the scheme generating the polytime computable functions the scheme of bounded recursion:

$$f(\bar{x}, \epsilon) = g(\bar{x})$$
$$f(\bar{x}, S(y)) = h(\bar{x}, y, f(y))|_{t(\bar{x},y)}$$

where S is the successr function in the lexicographic order.

The *classe of counting* (hierarchy of counting functions) is obtained by adding instead the (weaker) scheme of counting:

$$c(\bar{x}, \varepsilon) = \begin{cases} 0 & \text{if } f(\bar{x}, \epsilon) = 1 \\ \varepsilon & \text{otherwise} \end{cases}$$

$$c(\bar{x}, S(y)) = \begin{cases} S(c(\bar{x}, y)) & \text{if } f(\bar{x}, S(y)) = 1 \\ c(\bar{x}, y) & \text{otherwise} \end{cases}$$

Note

$c(\bar{x}, y) = \#\{w \leq_l y : f(\bar{x}, w) = 1\}.$

Second-order bounded variables

X^t, Y^q, Z^r: second-order bounded variables.

They have a *characteristic* axiom:

$$\forall X^t \forall y (y \in X^t \to y \leq t)$$

where y does not occur in the term t.

- The $\Sigma_0^{b,1}$-formulas constitute the smallest class of formulas containing the atomic formulas closed under bounded first-order quantifications. They define the (relativized) polytime hierarchy.
- A $\Sigma_1^{b,1}$-formula is a formula of the form $\exists X^t \phi(X^t)$, where ϕ is a $\Sigma_0^{b,1}$-formula. $\Pi_1^{b,1}$-formulas are defined dually.
- The second-order bounded formulas constitute the smallest class of formulas containing the atomic formulas and closed under first and second-order bounded quantifications.

Common axioms

- Basic fourteen axioms and characteristic axioms.

- Bounded comprehension for $\Sigma_0^{b,1}$-formulas ϕ:

$$\forall b \exists X^b \forall x \leq b (x \in X^b \leftrightarrow \phi(x)).$$

- Induction on notation for $\Sigma_0^{b,1}$-formulas. Ordinary induction for these formulas follows.

- Substitution scheme for $\Sigma_0^{b,1}$-formulas:

$$\forall x \leq b \exists X^z \phi(x, X^z) \to \exists Z^q \forall x \leq b \hat{\phi}(x, Z^q),$$

where q is a concretely presented term and $\hat{\phi}$ is obtained from ϕ by replacing $s \in X^z$ by $\langle x, s \rangle \in Z^q$.

U LISBOA UNIVERSIDADE DE LISBOA

The two second-order bounded theories

Definition

The theory $\Sigma_1^{b,1}$-NIA is the theory which adds to the commom axioms induction on notation for $\Sigma_1^{b,1}$-formulas.

Theorem

If $\Sigma_1^{b,1}$-NIA $\vdash \forall x \exists y\, \phi(x, y)$, where ϕ is a $\Sigma_1^{b,1}$-formula, then there is a polyspace description f such that $\forall x \phi(x, f(x))$.

Definition

The theory TCA (theory of counting arithmetic) is the theory which adds to the common axioms a counting axiom $\forall z \exists Z^q Count(Z^q, X^z)$, where q is concretely presented and $Count$ is a $\Sigma_0^{b,1}$-formula which expresses that Z^q is the graph of the function $x \rightsquigarrow \{x \leq_l z : x \in X^z\}$.

Theorem

If TCA $\vdash \forall x \exists y\, \phi(x, y)$, where ϕ is a $\Sigma_1^{b,1}$-formula, then there is a description of a counting function f such that $\forall x \phi(x, f(x))$.

ULISBOA UNIVERSIDADE DE LISBOA

Second-order bounded theories (continued)

Theorem
To either $\Sigma_1^{b,1}$-NIA *or* TCA*, we can add the scheme of collection for bounded second-order formulas and get a conservative extension with respect to sentences of the form* $\forall x \exists y \phi(x, y)$*, where* ϕ *is a bounded second-order formula.*

Lemma
The theory TCA *proves bounded comprehension for* $\Delta_1^{b,1}$*-formulas:*

$$\forall x \leq b \, (\phi(x) \leftrightarrow \varphi(x)) \rightarrow \exists X^b \forall x \, (x \in X^b \leftrightarrow \phi(x))$$

where ϕ *is a* $\Sigma_1^{b,1}$*-formula and* φ *is a* $\Pi_1^{b,1}$*-formula.*

Proof.
Let $\psi(x, X^1)$ be the biconditional $\phi(x) \leftrightarrow 1 \in X^1$. Note that ψ is $\Sigma_1^{b,1}$ and that TCA $\vdash \forall x \leq b \, \exists X^1 (\phi(x) \leftrightarrow 1 \in X^1)$. By substitution, one gets

$$\text{TCA} \vdash \exists Z^q \forall x \leq b \, (\phi(x) \leftrightarrow \langle x, 1 \rangle \in Z^q).$$

U LISBOA | UNIVERSIDADE DE LISBOA

The second-order theories

The *second-order* theories are framed in the language of second-order arithmetic. Second-order bounded variables are canonically interpreted in this language.

Definition

The theory BTPSA is the theory $\Sigma_1^{b,1}$-NIA together with the scheme of collection for bounded second-order formulas and the following recursive comprehension scheme:

$$\forall x\,(\exists y\phi(x,y) \leftrightarrow \forall z\varphi(x,z)) \rightarrow \exists X\forall x\,(x \in X \leftrightarrow \exists y\phi(x,y))$$

where ϕ is a $\exists\Sigma_1^{b,1}$-formula and φ is a $\forall\Pi_1^{b,1}$-formula.

The theory TCA^2 is as above, but starting with TCA.

Theorem

The theory BTPSA *(resp.* TCA^2*) is conservative over the theory* $\Sigma_1^{b,1}$-NIA *(resp.* TCA*) with the scheme of collection for bounded second-order formulas.*

The FAN_0 principle

Definition

The FAN_0 principle is the schema

$$\forall X \exists x \phi(x, X) \rightarrow \exists b \forall X \exists x \leq b\, \phi(x, X),$$

where ϕ a second-order bounded formula (possibly with parameters) in which b does not occur. The contrapositive of FAN_0 is known as *strict* Π^1_1*-reflection*.

Theorem

The theory $\mathsf{BTPSA}+\mathsf{FAN}_0$ *(resp.* $\mathsf{TCA}^2+\mathsf{FAN}_0$*) is conservative over* BTPSA *(resp.* TCA^2*) with respect to formulas without second-order unbounded quantifications.*

Proof.

A forcing argument *à la* Harrington, where the forcing conditions are infinite trees (defined by second-order bounded formulas) of bounded sets X^b (understood as encoding the "binary sequence" of its characteristic function).

References

S. Simpson: **Subsystems of Second Order Arithmetic**, Perspectives in Mathematical Logic, Springer-Verlag, 1999.

S. Buss: **Bounded Arithmetic**, Bibliopolis, Naples, 1986.

F. Ferreira: Polynomial Time Computable Arithmetic and Conservative Extensions. Ph.D. thesis, Pennsylvania State University, 1988.

F. Ferreira, "Polynomial time computable arithmetic." In: *Logic and Computation*, W. Sieg (editor), Contemporary Mathematics 106, American Mathematical Society 1990, 137-156.

References

S. Buss, "A conservation result concerning bounded theories and the collection axiom." *Proc. Amer. Math. Soc.* 100, 709-716 (1987).

F. Ferreira, "A note on a result of Buss concerning bounded theories and the collection scheme." *Portugaliae Mathematica* 52, 331-336 (1995).

F. Ferreira, "A feasible theory for analysis." **The Journal of Symbolic Logic** 59, 1001-1011 (1994)

A. Fernandes, F. Ferreira & G. Ferreira, "Techniques in weak analysis for conservation results." In: *New Studies in Weak Arithmetics*, P. Cégielski, Ch. Cornaros and C. Dimitracopoulos (eds.), CSLI Publications (Stanford) and Presses Universitaires (Paris 12) 2013, 115-147.

References

G. Ferreira: Aritmética Computável em Espaço Polinomial. Master thesis, Universidade de Lisboa, 2001.

R. Bianconi, G. Ferreira & E. Silva, “Bounded theories for polyspace computability.” **Portugaliae Mathematica** 70, 295-318 (2013).

J. Johannsen & C. Pollett, “On proofs about threshold circuits and counting hierarchies.” In: Thirteenth Annual IEEE Symposium on Logic in Computer Science, IEEE Press 1998,, 444-452.

G. Ferreira: Sistemas de Análise Fraca para Integração. Ph.D. thesis, Universidade de Lisboa, 2006.

References

J. Barwise: **Admissible Sets and Structures**. Perspectives in Mathematical Logic, Springer-Verlag, 1975.

A. Cantini, “Asymmetric interpretations for bounded theories.” **Mathematical Logic Quarterly** 42, 270-288 (1996).

A. Fernandes: Investigações em Sistemas de Análise Exequível. Ph.D. thesis, Universidade de Lisboa, 2001.

A. Fernandes, “Strict Π^1_1-reflection in bounded arithmetic.” **Archive for Mathematical Logic** 49, 17-34 (2009).

References

S. Cook & A. Urquhart, “Functional interpretations of feasibly constructive arithmetic.” **Annals of Pure and Applied Logic** 63, 103-200 (1993).

F. Ferreira & P. Oliva, “Bounded functional interpretation and feasible analysis.” **Annals of Pure and Applied Logic** 145, 115-129 (2007).

To be continued