Weak Analysis: Mathematics

Fernando Ferreira

Universidade de Lisboa

Journées sur les Arithmétiques Faibles 33
University of Gothenburg

June 16-18, 2014

Simple consequences of recursive comprehension

A function $f : X \mapsto Y$ is given by a set of ordered pairs. We can state $f(x) \in Z$ in two ways:

$$x \in X \wedge \exists y(\langle x, y\rangle \in f \wedge y \in Z)$$

$$x \in X \wedge \forall y(\langle x, y\rangle \in f \rightarrow y \in Z)$$

Thus, $\{x \in X : f(x) \in Z\}$ exists in BTFA.

Similar thing for the composition of two functions.

Proposition

The theory BTFA *proves the* $\exists\Sigma^b_1$*-path comprehension scheme, i.e.,*

$$Path(\phi_x) \rightarrow \exists X \forall x(\phi(x) \leftrightarrow x \in X),$$

where ϕ *is* $\exists\Sigma^b_1$*-formula.*

Proof.

$\phi(x)$ is equivalent to $\forall y(y \equiv x \wedge y \neq x \rightarrow \neg\phi(x))$.

Polytime arithmetic in Σ_1^b-NIA

- $\mathbb{N}_1$: tally numbers (elements u such that $u = 1 \times u$). Model of $I\Delta_0$, but one can also make definitions by bounded recursion on the tally part.

- $\mathbb{N}_2$: dyadic natural numbers of the form $1w$ or ε, where w is a binary string ($w \in \mathbb{W}$). Polytime arithmetic.

- $\mathbb{D}$: dyadic rational numbers. Have the form $\langle \pm, x, y \rangle$, where x (resp. y) is ε or a binary string starting with 1 (resp. ending with 1). Dense ordered ring without extremes.

- Given $n \in \mathbb{N}_1$, 2^n is $\langle +, 1\underbrace{00\ldots0}_{n \text{ zeros}}, \epsilon \rangle$; 2^{-n} is $\langle +, \epsilon, \underbrace{00\ldots0}_{n-1 \text{ zeros}}1 \rangle$.

- $\mathbb{D}$ is not a field but it is always closed by divisions by tally powers of 2.

Real numbers in BTFA

Definition

We say that a function $\alpha : \mathbb{N}_1 \mapsto \mathbb{D}$ is a *real number* if $|\alpha(n) - \alpha(m)| \leq 2^{-n}$ for all $n \leq m$. Two real numbers α and β are said to be *equal*, and we write $\alpha = \beta$, if $\forall n \in \mathbb{N}_1 |\alpha(n) - \beta(n)| \leq 2^{-n+1}$.

The real number system is an ordered field. The relations $\alpha = \beta$, $\alpha \leq \beta$, $\alpha + \beta \leq \gamma$, ... are $\forall\Pi_1^b$-formulas, while $\alpha \neq \beta$, $\alpha < \beta$, ... are $\exists\Sigma_1^b$-formulas.

A *dyadic real number* is a triple of the form $\langle \pm, x, X \rangle$ where $x \in \mathbb{N}_2$ and X is an infinite path.

Continuous partial functions

Definition

Within BTFA, a (code for a) *continuous partial function* from $\mathbb{R}$ into $\mathbb{R}$ is a set of quintuples $\Phi \subseteq \mathbb{W} \times \mathbb{D} \times \mathbb{N}_1 \times \mathbb{D} \times \mathbb{N}_1$ such that:

1. if $\langle x, n\rangle \Phi \langle y, k\rangle$ and $\langle x, n\rangle \Phi \langle y', k'\rangle$, then $|y - y'| \leq 2^{-k} + 2^{-k'}$;
2. if $\langle x, n\rangle \Phi \langle y, k\rangle$ and $\langle x', n'\rangle < \langle x, n\rangle$, then $\langle x', n'\rangle \Phi \langle y, k\rangle$;
3. if $\langle x, n\rangle \Phi \langle y, k\rangle$ and $\langle y, k\rangle < \langle y', k'\rangle$, then $\langle x, n\rangle \Phi \langle y', k'\rangle$;

where $\langle x, n\rangle \Phi \langle y, k\rangle$ stands for $\exists\Sigma_1^b$-relation $\exists w \, \langle w, x, n, y, k\rangle \in \Phi$, and where $\langle x', n'\rangle < \langle x, n\rangle$ means that $|x - x'| + 2^{-n'} < 2^{-n}$.

Definition

Let Φ be a continuous partial real function of a real variable. We say that a real number α is in the *domain* of Φ if

$$\forall k \in \mathbb{N}_1 \exists n \in \mathbb{N}_1 \exists x, y \in \mathbb{D} \left(|\alpha - x| < 2^{-n} \wedge \langle x, n\rangle \Phi \langle y, k\rangle\right).$$

Continuous functions (continued)

Definition

Let Φ be a continuous partial real function and let α be a real number in the domain of Φ. We say that a real number β is the *value of* α *under the function* Φ, and write $\Phi(\alpha) = \beta$, if

$$\forall x, y \in \mathbb{D} \, \forall n, k \in \mathbb{N}_1 \, [\langle x, n \rangle \Phi \langle y, k \rangle \wedge |\alpha - x| < 2^{-n} \to |\beta - y| \leq 2^{-k}].$$

Note

$\Phi(\alpha) = \beta$ *is a* $\forall \Pi^b_1$ *notion. Etc.*

Theorem (BTFA)

Let Φ *be a continuous partial real function and let* α *in the domain of* Φ*. Then there is a dyadic real number* β *such that* $\Phi(\alpha) = \beta$*. Moreover, this real number is unique.*

Corollary

Every real number can be put in dyadic form.

Intermediate value theorem

Theorem (BTFA)

If Φ *is a continuous function which is total in the closed interval* $[0,1]$ *and if* $\Phi(0) < 0 < \Phi(1)$*, then there is a real number* $\alpha \in [0,1]$ *such that* $\Phi(\alpha) = 0$*.*

Proof.

Assume that there is no dyadic rational number $x \in [0,1]$ such that $\Phi(x) = 0$. Consider $X := \{x : x \in \mathbb{D} \cap [0,1] \wedge \Phi(x) < 0\}$ (it exists!).

Define by bounded recursion along the tally part, the function $f : \mathbb{N}_1 \to \mathbb{D} \times \mathbb{D}$ according to the clauses $f(0) = \langle 0, 1 \rangle$ and

$$f(n+1) = \begin{cases} \langle (f_0(n) + f_1(n))/2, f_1(n) \rangle & \text{if } (f_0(n) + f_1(n))/2 \in X \\ \langle f_0(n), (f_0(n) + f_1(n))/2 \rangle & \text{otherwise} \end{cases}$$

where f_0 and f_1 are the first and second projections of f. These projections determine the same real α, and $\Phi(\alpha) = 0$.

ULISBOA UNIVERSIDADE DE LISBOA

Real closed ordered fields

- The real system constitutes a real closed ordered field.
- Can define polynomials of tally degree as functions $F : \{i \in \mathbb{N}_1 : i \leq d\} \times \mathbb{N}_1 \to \mathbb{D}$ such that, for every $i \leq d$, the function γ_i defined by $\gamma_i(n) = F(i, n)$ is a real number.
- Given $P(X) = \gamma_d X^d + \cdots + \gamma_1 X + \gamma_0$ can define it as a continuous function.
- Generalize to series. Can introduce some transcendental functions. This has not been worked out.

The Heine-Borel theorem

Definition (BTFA)

A (code for an) open set U is a set $U \subseteq \mathbb{W} \times \mathbb{D} \times \mathbb{N}_1$. We say that a real number α is an *element* of U, and write $\alpha \in U$, if

$$\exists z \in \mathbb{D} \exists n \in \mathbb{N}_1 (|\alpha - z| < 2^{-n} \wedge \exists w \langle w, z, n \rangle \in U).$$

Suppose that U is an open set and that $[0,1] \subseteq U$. The *Heine-Borel theorem* states the existence of $k \in \mathbb{N}_1$ such that, for all $\alpha \in [0,1]$,

$$\exists z \in \mathbb{D}, n \in \mathbb{N}_1, w \in \mathbb{W} \, (z, n, w \leq k \ \wedge |\alpha - z| < 2^{-n} \wedge \langle w, z, n \rangle \in U).$$

Theorem (BTFA)

The Heine/Borel theorem for $[0,1]$ *is equivalent to* Π^b_1-WKL.

The uniform continuity theorem

Definition

Let $\Phi : [0,1] \mapsto \mathbb{R}$ be a (total) continuous function. We say that Φ is *uniformly continuous* if

$$\forall k \in \mathbb{N}_1 \exists m \in \mathbb{N}_1 \forall \alpha, \beta \in [0,1](|\alpha - \beta| \leq 2^{-m} \to |\Phi(\alpha) - \Phi(\beta)| < 2^{-k}).$$

Theorem (BTFA)

The principle that every (total) real valued continuous function defined on $[0,1]$ is uniformly continuous implies WKL *and is implied by* Π_1^b-WKL.

Proof.

The latter statement uses the Heine-Borel theorem. The former statement uses the following lemma:

Lemma (BTFA)

Let T be a subtree of $\mathbb{W}$ with no infinite paths. There is a continuous (total) function defined on $[0,1]$ such that, for all end nodes x of T, $\Phi(.x^*) = 2^{l(x)}$ (where $l(x)$ is the unary length of x).

The attainement of maximum

The Σ_1^b-IA induction principle is

$$\phi(\epsilon) \wedge \forall x(\phi(x) \to \phi(S(x))) \to \forall x \phi(x),$$

for $\phi \in \Sigma_1^b$.

Note

Over BTFA, *the* Σ_1^b-IA *induction principle is equivalent to saying that every non-empty bounded set X of $\mathbb{W}$ has a lexicographic maximum (minimum).*

Theorem

Over BTFA + Σ_0-WKL, the following are equivalent:

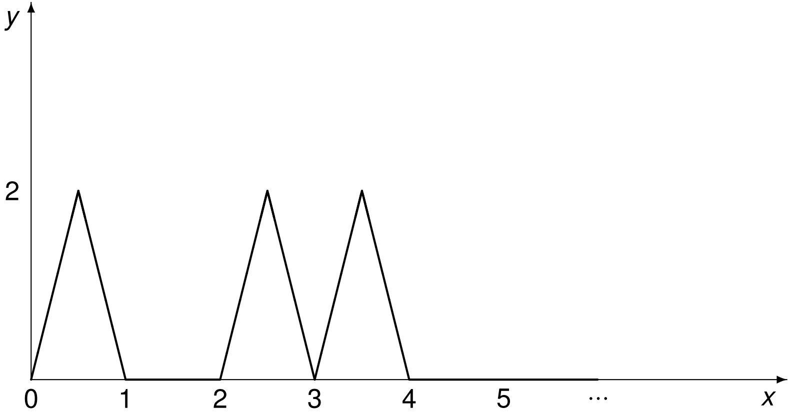
(a) Every (total) real valued continuous function defined on $[0, 1]$ has a maximum.

(b) Every (total) real valued continuous function defined on $[0, 1]$ has a supremum.

(c) Σ_1^b-IA.

Integration and counting

Given $X \subseteq \mathbb{N}_2$ a non-empty subset, let Φ_X be the continuous function



where a "spike" follows x exactly when $x \in X$.

Integration and counting (continued)

The counting function f up to b is given simultaneously by:

$$f = \{\langle x, n\rangle : x, n \in \mathbb{N}_2, x \leq b+1,\ \int_0^x \Phi_X(t)dt =_{\mathbb{R}} n_{\mathbb{R}}\}$$

$$f = \{\langle x, n\rangle : x, n \in \mathbb{N}_2, x \leq b+1,\ n - \tfrac{1}{2} <_{\mathbb{R}} \int_0^x \Phi_X(t)dt <_{\mathbb{R}} n_{\mathbb{R}} + \tfrac{1}{2}\}$$

Get f by (the) recursive comprehension (available in BTFA).

How does one prove that the two above definitions coincide? Can we show that, for each $x \leq b$, $\int_0^x \Phi_X(t)dt$ is equal to a (dyadic) natural number. By induction? *Prima facie*, we do have have this kind of induction!

1. The unbounded quantifiers can be dealt by judicious uses of bounded collection.

2. Σ_1^b-IA induction is available because we can prove that every non-empty set has a minimum. Use the intermediate value theorem!

Counting and integration

- If we can count, then we can add:

$$\sum_{w=0}^{x} f(w) = \#\{u : \exists w \leq_2 x \exists y <_2 f(w)\, (u = \langle w, y \rangle)\}.$$

Definition

Let Φ be a continuous total function on [0,1]. A *modulus of uniform continuity (m.u.c)* is a strictly increasing function $h : \mathbb{N}_1 \mapsto \mathbb{N}_1$ such that

$$\forall n \in \mathbb{N}_1 \forall \alpha, \beta \in [0,1](|\alpha - \beta| \leq 2^{-h(n)} \to |\Phi(\alpha) - \Phi(\beta)| < 2^{-n}).$$

Note

Over $\mathrm{TCA}^2 + \Sigma_0$-WKL, *if* Φ *is a continuous total function on* [0,1] *then* Φ *has a m.u.c.*

Counting and integration (continued)

Definition (TCA^2)

Take Φ a continuous total function on [0,1] with a m.u.c. h. The *integral of* Φ *between 0 and 1* is defined by

$$\int_0^1 \Phi(t)dt :=_{\mathbb{R}} \lim_n S_n.$$

where, for all $n \in \mathbb{N}_1$, $S_n = \sum_{w=0}^{2^{h(n)}-1} \frac{1}{2^{h(n)}} \Phi(\frac{w}{2^{h(n)}}, n)$. Here $\Phi(r, n)$ is a suitable approximation of $\Phi(r)$.

Note

The above definition readily extends to integration for intervals with dyadic rational points as limits.

Let $d : \mathbb{D} \mapsto \mathbb{D}$ be:

$$d(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \end{cases}$$

The fundamental theorem of calculus

Given Φ a continuous total function on [0,1] with m.u.c. h, define $\langle x, n\rangle \Psi \langle y, k\rangle$ as follows:

$$x, y \in \mathbb{D} \wedge n, k \in \mathbb{N}_1 \wedge \left| \int_0^{d(x)} \Phi(t)dt \right| < \frac{1}{2^k} - \frac{1}{2^{n-m-1}},$$

where $m \in \mathbb{N}_1$ is such that $\forall \alpha \in [0,1]\, |\Phi(\alpha)| \leq 2^m$.

- The above Ψ gives, within TCA^2, the definition of the continuous real function $\alpha \rightsquigarrow \int_0^\alpha \Phi(t)dt$.
- It is easy to prove that the derivative of Ψ at α is $\Phi(\alpha)$.

On continuous functions

- Takeshi Yamazaki defined continuity via uniform approximations of piecewise linear functions. Uniform continuity is built in.
- What about defining continuity via uniform approximations of polynomials? Do we get a nice theory of integration in BTFA?
- Weierstrass' approximation theorem: every (uniformly) continuous function on [0,1] is uniformly approximated by polynomials.
- Conjecture. Over BTFA (or close enough), Weierstrass' approximation theorem is equivalent to the totality of exponentiation.

Interpretability in Robinson's Q

- The theories $I\Delta_0 + \Omega_n$ are interpretable in Robinson's Q.
- Ω_{n+1} means that the logarithmic part satisfies Ω_n.
- The RSUV isomorphism characterizes the theory of the logarithmic part of a model (and vice-versa).
- Hence, lots of interpretability in Q. Basically, it includes any computations that take a (fixed) iterated exponential number of steps. The "fixed" is for the number of iterations.
- Note that $I\Delta_0 + \exp$ is <u>not</u> interpretable in Q.

Interpretability in Robinson's Q (continued)

Theorem

The theory BTFA *is interpretable in Robinson's* Q.

Proof.

Let $U(e, x, y, p, c)$ be a 5-ary sw.q.-formula with the universal property according to which, for every ternary sw.q.-formula $\psi(x, y, p)$, there is a (standard) binary string e such that

$$\Sigma_1^b\text{-NIA} \vdash \forall x \forall y \forall p(\psi(x, y, p) \leftrightarrow \exists c\, U(e, x, y, p, c)).$$

Define

$$Set(\alpha) := \forall x(\exists w U(\alpha_0, x, w_0, \alpha_1, w_1) \leftrightarrow \forall w \neg U(\alpha_2, x, w_0, \alpha_3, w_1)),$$

where α is seen as the quadruple $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$. □

Corollary

Tarski's theory of real closed ordered fields is interpretable in Q.

- BTPSA is interpretable in Q. Can get more than that! Question: Can we add (suitable versions of) weak König's lemma and still get interpretability in Q?

U LISBOA | UNIVERSIDADE DE LISBOA

References

A. Fernandes & F. Ferreira, “Groundwork for weak analysis.” **The Journal of Symbolic Logic** 67, 557-578 (2002).

A. Fernandes & F. Ferreira, “Basic applications of weak König’s lemma in feasible analysis.” In: *Reverse Mathematics 2001*, S. Simpson (editor), Association for Symbolic Logic / A K Peters 2005, 175-188.

F. Ferreira & Gilda Ferreira, “Counting as integration in feasible analysis.” **Mathematical Logic Quarterly** 52, 315-320 (2006).

F. Ferreira & G. Ferreira, “The Riemann integral in weak systems of analysis.” **Journal of Universal Computer Science** 14, 908-937 (2008).

References

T. Yamazaki, "Reverse mathematics and weak systems of 0-1 for feasible analysis." In: *Reverse Mathematics 2001*, S. Simpson (editor), Association for Symbolic Logic / A K Peters 2005, 394-401.

A. Fernandes, "The Baire category theorem over a feasible base theory." In: *Reverse Mathematics 2001*, S. Simpson (editor), Association for Symbolic Logic / A K Peters 2005, 164-174.

E. Nelson: **Predicative Arithmetic**. Mathematical Notes, Princeton University Press, 1986.

Hajek, Pudlák: **Metamathematics of First-order Arithmetic**. Perspectives in Mathematical Logic, Springer-Verlag, 1993.

F. Ferreira & G. Ferreira, "Interpretability in Robinson's Q." **The Bulletin of Symbolic Logic** 19, 289-317 (2013).

Thank you