The problem of a model with no collection and no exponentiation

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Dramatis personae

$I\Delta_0$:

induction for bounded formulas in language of ordered rings.

$B\Sigma_1$:

$\forall x{<}v\,\exists y\,\psi(x,y) \Rightarrow \exists w\,\forall x{<}v\,\exists y{<}w\,\psi(x,y)$, for ψ bounded.

Exp:

$\forall x\,\exists 2^x$.

Ω_1:

$\forall x\,\exists\omega_1(x)$, where $\omega_1(x) = x^{\log x}$.

Basic facts—and a question

For T reasonable and Π_1-axiomatized, $T \preceq_{\Pi_2} T + B\Sigma_1$ (Buss 1987).

On the other hand, $\Pi_2(\mathbb{N}) \nvdash B\Sigma_1$ (Parsons 1970).
But, all proofs of this make use of exponentially large objects.
(Or larger.)

Hence, a question (Wilkie-Paris 1989):

$$\text{Does } I\Delta_0 + \neg\mathrm{Exp} \vdash B\Sigma_1?$$

Example proof of $\mathrm{I}\Delta_0 \not\vdash \mathrm{B}\Sigma_1$

- Take $\mathcal{M} \models \mathrm{I}\Delta_0 + \mathrm{Exp}$, $a \in \mathcal{M} \setminus \mathbb{N}$.
- $\mathcal{K}_1(\mathcal{M})$ consists of $\Sigma_1(a)$-definable elements of $\mathcal{M}$.
- $\mathcal{K}_1(\mathcal{M})$ satisfies $\forall x < a \, \exists \varphi < \log a$ Σ_1-def'n such that $\mathrm{Sat}(\varphi, x)$.
- But the $\exists$ quantifier in Sat can't be bounded.

(Kirby-Paris 1978.)

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Exp is only needed to have the formula Sat.
If there is a model of $I\Delta_0 + \neg\text{Exp}$ with a Σ_1 universal formula,
then $I\Delta_0 + \neg\text{Exp} \nvdash B\Sigma_1$.

Other proofs

Some other proofs have a proof-theoretic/computational character.

If $\Pi_2(\mathbb{N}) \vdash \mathrm{B}\Sigma_1$, then...

- Π_1 relations of a certain class have Σ_1 “almost uniformizations” (Adamowicz 1988),
- for function f with elementary recursive graph, the closure of f and elementary functions under composition is also closed under bounded max (Beklemishev 1998),

but neither of these things actually happens.

Conditional proofs of $\mathrm{I}\Delta_0 + \neg\exp \nvdash \mathrm{B}\Sigma_1$

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Conditional proofs of $I\Delta_0 + \neg\exp \nvdash B\Sigma_1$

We know that $I\Delta_0 + \neg\exp \nvdash B\Sigma_1$ if...

1. ... the polynomial-time hierarchy collapses.
   - Reason: we then have a model of $\Pi_1(\mathbb{N}) + \neg\mathrm{Exp}$ with a Σ_1 universal formula.

Conditional proofs of $I\Delta_0 + \neg\exp \nvdash B\Sigma_1$

We know that $I\Delta_0 + \neg\exp \nvdash B\Sigma_1$ if...

2. ... the size parameter in a Δ_0 truth definition for Δ_0 formulas about a has to be above $2^{a^{\mathbb{N}}}$ (Paris 1980's, only published in AKP 2012).

- Basically says that the obvious deterministic algorithm for evaluating a formula in a finite structure sometimes cannot be significantly improved even by a Σ_k procedure for $k > 0$.
- Inconsistent with collapse of PH.

Conditional proofs of $\mathrm{I}\Delta_0 + \neg\exp \nvdash \mathrm{B}\Sigma_1$

We know that $\mathrm{I}\Delta_0 + \neg\exp \nvdash \mathrm{B}\Sigma_1$ if...

3. ... exists elementary recursive f with Δ_0 graph such that closure of f and linear-time hierarchy functions under composition is not closed under bounded max (Cordón Franco et al. 2014).

   - Condition in the spirit of Beklemishev 1998.

Conditional proofs of $I\Delta_0 + \neg\exp \nvdash B\Sigma_1$

We know that $I\Delta_0 + \neg\exp \nvdash B\Sigma_1$ if...

4. ... there is $\mathcal{M} \models I\Delta_0 + \neg\exp$ with cofinal Σ_1-definable elements and Σ_1 truth definition for Σ_1 sentences (AKP 2012).

- Proof by simple compactness argument and Kirby-Paris.
- Intriguing, because...

Truth definitions for sentences

Theorem (AKP 2012)

There is $\mathcal{M} \models \mathrm{I}\Delta_0 + \neg\mathrm{Exp}$ *with cofinal* Σ_1 *definable elements and a* Σ_2 *truth definition for* Σ_2 *sentences.*

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Proof.

- build theory T by deciding for each Σ_1 sentence $\varphi_0, \varphi_1, \dots$ whether it is true or false,
- φ_0 is true (according to T) and inconsistent with Exp,
- $\neg\varphi_{n+1}$ is true unless inconsistent with previous choices.
- Σ_1 sentence is equivalent in T to an “inconsistency statement” (bool. comb. of Σ_1 sentences with simple bounded parts),
- in suitable model, this gives Σ_2 truth definition for Σ_2 sentences.

□

So...

If we can show $\mathrm{I}\Delta_0 + \neg\exp \nvdash \mathrm{B}\Sigma_1$ in so many different cases, why can't we prove it outright?

What case is left out?

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If we can show $\mathrm{I}\Delta_0 + \neg\exp \nvdash \mathrm{B}\Sigma_1$ in so many different cases, why can't we prove it outright?

What case is left out?

Both PH $\downarrow$ and Paris' condition imply something like: even if $m \gg \ell$, sometimes Σ_m computations are no faster than Σ_ℓ computations.

So the difficult case seems to be if we can always decrease time at the cost of adding more quantifier alternations.

Aha!

- Known situations like “more of a weaker resource can be simulated by less of a stronger resource” seem to have something do to with end-extensions (Ferreira 1996, Zambella 1997).
- End-extensions obviously have something to do with $B\Sigma_1$: a model of $I\Delta_0$ with an end-extension always satisfies $B\Sigma_1$.

A theorem about $\neg\Omega_1$

Reminder:

- Δ_0 formulas $\leftrightsquigarrow$ linear-time hierarchy (LinH),
- bounded formulas with ω_1 $\leftrightsquigarrow$ polynomial-time hierarchy (PH).
- "PH $=$ LinH?" is open, and probably very hard.

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- bounded formulas with ω_1 $\leftrightsquigarrow$ polynomial-time hierarchy (PH).
- "PH = LinH?" is open, and probably very hard.

Theorem

If PH = LinH, *then* $\Pi_1(\mathbb{N}) + \neg\Omega_1 \vdash \mathrm{B}\Sigma_1$.

A theorem about $\neg\Omega_1$ (cont'd)

Theorem

If PH $=$ LinH, *then* $\Pi_1(\mathbb{N}) + \neg\Omega_1 \vdash \mathrm{B}\Sigma_1$.

Proof.

- Let $\mathcal{M} \models \Pi_1(\mathbb{N}) + \neg\Omega_1$, $\mathcal{M} \preccurlyeq_{\Delta_0} \mathcal{N} \models \mathrm{Th}(\mathbb{N})$.
- Let $\mathcal{K}$ = closure of $\mathcal{M}$ in $\mathcal{N}$ under Skolem functions for PH properties.
- Always, $\mathcal{M} \subseteq \mathcal{K} \models \Pi_1(\mathbb{N}) + \Omega_1$.
- But if PH $=$ LinH, then $\mathcal{K}$ end-extends $\mathcal{M}$!

□

A version about $I\Delta_0$

Theorem

Assume there is a translation $\psi \mapsto \psi^{\mathrm{lin}}$ of bounded f'las with ω_1 to Δ_0 f'las such that $\mathrm{I}\Delta_0 + \Omega_1 + \{\forall x\,(\psi(x) \Leftrightarrow \psi^{\mathrm{lin}}(x)) : \psi \text{ bounded}\}$ is Π_1-conservative over $I\Delta_0$.
Then $\mathrm{I}\Delta_0 + \neg\Omega_1 \vdash \mathrm{B}\Sigma_1$.

Remark:
The assumption is a reasonable way of saying $\mathrm{I}\Delta_0 \vdash \mathrm{PH} = \mathrm{LinH}$.

Question:
Does this follow from "$\mathrm{I}\Delta_0 + \Omega_1$ is Π_1-conservative over $I\Delta_0$"?

What about ¬Exp?

- It would be nice to do the same with PH replaced by $\mathrm{EH} = \bigcup_{k\in\mathbb{N}} \Sigma_k\text{-TIME}(2^{O(n)})$.

What about $\neg$Exp?

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- However, $\mathrm{EH} \neq \mathrm{LinH}$ is known!
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- However, $\mathrm{EH} \neq \mathrm{LinH}$ is known!
- Same remains true for $\bigcup_{k\in\mathbb{N}} \Sigma_k\text{-TIME}(f)$ where f is *fractional-exponential* (finite iteration of f dominates 2^x).
- But refuting the following seems beyond reach: "for every k there is fractional-exponential f such that $\Sigma_k\text{-TIME}(f) \subseteq \mathrm{LinH}$".

A theorem about $\neg$Exp

Theorem

Assume that for every k there is fractional-exponential f such that $\bigcup_{k \in \mathbb{N}} \Sigma_k\text{-TIME}(f^{O(1)}) \subseteq \text{LinH}$. *Then* $\Pi_1(\mathbb{N}) + \neg\text{Exp} \vdash \text{B}\Sigma_1$.

- The proof is similar to the one for $\neg\Omega_1$, but the end-extensions are now to models of finite fragments of $\text{I}\Delta_0 + \Omega_1$ (Buss' T^k_2's).
- There is a version about $\text{I}\Delta_0 + \neg\,\text{exp}$.
  It won't fit on the remainder of this slide.

Relativizations

Can we use the previous results to show that any proof of $I\Delta_0 + \neg\mathrm{Exp} \nvdash B\Sigma_1$ has to be "non-relativizing"?

How would we even express that?

Relativizations

Can we use the previous results to show that any proof of $I\Delta_0 + \neg\mathrm{Exp} \nvdash B\Sigma_1$ has to be "non-relativizing"?

How would we even express that?

Conjecture

Let α be a new unary relation symbol. There exists a consistent recursively axiomatized set of $\Pi_1(\alpha)$ sentences $\mathrm{T}(\alpha)$ such that $\mathrm{T}(\alpha) + \neg\mathrm{Exp} \vdash B\Sigma_1(\alpha)$.

A relativized result

Theorem

Let α be a new unary relation symbol. For every finite fragment $\mathrm{B}(\alpha)$ *of* $\mathrm{B}\Sigma_1(\alpha)$ *there exists a consistent recursively axiomatized set of* $\Pi_1(\alpha)$ *sentences* $\mathrm{T_B}(\alpha)$ *such that* $\mathrm{T_B}(\alpha) + \neg\mathrm{Exp} \vdash \mathrm{B}(\alpha)$.

- This is not true without $\neg$Exp.
  E.g. Kirby-Paris argument relativizes.
- Proof uses variant of "Håstad's second switching lemma" and known construction of oracle for $\mathrm{E} \subseteq \mathrm{LinH}$ to show that every finite level of EH can be put inside LinH relative to an oracle. (Oracle has to depend on level since $\mathrm{EH} \neq \mathrm{LinH}$ relativizes.)