Local Induction Axioms vs Local Induction Rules

A. Cordón–Franco,
F. F. Lara–Martín

University of Seville (Spain)
Partially supported by grant MTM2011-26840 of MICINN, Spanish Goverment

33ème Journées sur les Arithmétiques Faibles
University of Gothenburg, June 2014

Contents

Classical Induction Axioms

- Peano Arithmetic is axiomatized over a basic theory (say, Robinson's Q theory) by the induction scheme:

$$I_{\varphi,x}: \quad \varphi(0,v) \wedge \forall x\,(\varphi(x,v) \rightarrow \varphi(x+1,v)) \rightarrow \forall x\, \varphi(x,v)$$

- Classical fragments:

$$I\Sigma_n = Q + \{I_{\varphi,x} : \ \varphi(x,v) \in \Sigma_n\}$$
$$I\Pi_n = Q + \{I_{\varphi,x} : \ \varphi(x,v) \in \Pi_n\}$$

- Well known fact: $I\Sigma_n \equiv I\Pi_n$.
- This equivalence fails for **Parameter free schemes**.
  - We write $\varphi(x) \in \Sigma_n^-$ if $\varphi(x) \in \Sigma_n$ and x is the only free variable of $\varphi(x)$.
  - $I\Sigma_n^- = Q + \{I_{\varphi,x} : \ \varphi(x) \in \Sigma_n^-\}$.
  - $I\Pi_n^-$ is defined accordingly.
- $(n \geq 1) \quad I\Sigma_n^-$ is a proper extension of $I\Pi_n^-$.

Induction Rules

Let $\Gamma = \Sigma_n$ (Π_n or $\mathcal{B}(\Sigma_n)$), and let T be a theory, $I\Delta_0 \subseteq T$.

- Γ–IR is the inference rule given by

$$\frac{\varphi(0,v) \wedge \forall x\,(\varphi(x,v) \rightarrow \varphi(x+1,v))}{\forall x\,\varphi(x,v)}, \quad \varphi(x,v) \in \Gamma.$$

- Γ–IR_0 denotes the inference rule

$$\frac{\forall x\,(\varphi(x,v) \rightarrow \varphi(x+1,v))}{\varphi(0,v) \rightarrow \forall x\,\varphi(x,v)}, \quad \varphi(x,v) \in \Gamma.$$

- If R is an inference rule then
  - $[T,R]$ denotes the closure of T under first order logic and unnested applications of R.
  - $T+R$ denotes the closure of T under first order logic and (nested) applications of R.
  - $[T,R]_0 = T, \quad [T,R]_{m+1} = [[T,R]_m, R]$.
- Γ^-–IR (resp. Γ^-–IR_0) denotes the parameter free version of Γ–IR (resp. Γ–IR_0).

Some basic results

- Some basic relations:

$$[T, \Sigma_1\text{–IR}] \equiv [T, \Sigma_1\text{–IR}_0] \equiv [T, \Sigma_1^-\text{–IR}] \equiv [T, \Pi_1\text{–IR}_0].$$

- (Parsons) $I\Sigma_1$ is Π_2–conservative over $I\Delta_0 + \Sigma_1$–IR.
- (Adamowicz–Bigorajska; Mints; Ratajczyk; Kaye) For every $m \geq 1$, if $\varphi_1(x), \dots, \varphi_m(x) \in \Sigma_1^-$ and $\theta \in \Pi_2$

$$I\Delta_0 + I_{\varphi_1} + \cdots + I_{\varphi_m} \vdash \theta \quad \Rightarrow \quad [I\Delta_0, \Sigma_1\text{–IR}]_m \vdash \theta$$

- There is no nontrivial conservation between $I\Sigma_1$ and $I\Delta_0 + \Pi_1$–IR.
- $[I\Delta_0, \Pi_1\text{–IR}] \subset [I\Delta_0, \Pi_1^-\text{–IR}_0] \subset [I\Delta_0, \Pi_1\text{–IR}_0]$.

Reflection principles

- We work over $\mathrm{EA} = I\Delta_0 + \exp$.
- For each theory T, recursively axiomatizable, we consider formulas
  - $\mathrm{Prf}_T(y, x)$ expresing "y is (codes) a proof of x in T"
  - $\mathrm{Prov}_T(x) \equiv \exists y\, \mathrm{Prf}_T(y, x)$
- Local Reflection for T is the following scheme, $\mathrm{Rfn}(T)$,

$$\mathrm{Prov}_T(\ulcorner \varphi \urcorner) \rightarrow \varphi$$

  for each sentence φ.
- Partial Local Reflection, $\mathrm{Rfn}_\Gamma(T)$ is given by

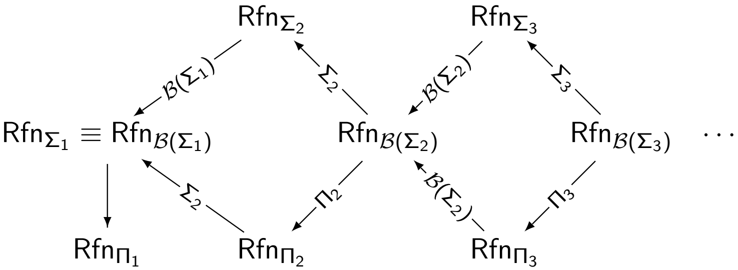
$$Prov_T(\ulcorner \varphi \urcorner) \rightarrow \varphi$$

  for every $\varphi \in \Gamma \cap \mathrm{Sent}$. Here $\Gamma = \Sigma_n,\ \Pi_n$ or $\mathcal{B}(\Sigma_n)$ $(n \geq 1)$.

Conservation for Local Reflection

Theorem. (Beklemishev)
Let $\Gamma = \Sigma_n$ or Π_n with $n \geq 2$ or $\Gamma = \mathcal{B}(\Sigma_k)$, with $k \geq 1$, then

- $T + \mathrm{Rfn}(T)$ is Γ–conservative over $T + \mathrm{Rfn}_\Gamma(T)$.



A stronger conservation result

Notation: If Φ is a set of sentences and $m \geq 1$, we write

$$T + \Phi \vdash_m \theta$$

to express that θ is derivable using axioms form T and at most m sentences in Φ.

Theorem. (Beklemishev)
Let $\Gamma = \Sigma_n$ or Π_n with $n \geq 2$ or $\Gamma = \mathcal{B}(\Sigma_k)$, with $k \geq 1$, then for every $m \geq 1$,

- For all $\theta \in \Gamma \cap \text{Sent}$,

$$\text{If } T + \text{Rfn}(T) \vdash_m \theta \quad \text{then} \quad T + \text{Rfn}_\Gamma(T) \vdash_m \theta$$

- Let $T_0 = T$ and $T_{j+1} = T_j + \text{Con}(T_j)$, then, for every $\theta \in \Pi_1 \cap \text{Sent}$

$$\text{If } T + \text{Rfn}(T) \vdash_m \theta \quad \text{then} \quad T_m \vdash \theta$$

Some results á la Kreisel–Lévy

- (Kreisel–Lévy) $\mathsf{PA} \equiv \mathsf{EA} + \mathsf{RFN}(\mathsf{EA})$.
- (Leivant-Ono) For $(n \geq 1)$

$$\mathsf{I}\Sigma_n \equiv \mathsf{EA} + \mathsf{RFN}_{\Sigma_{n+1}}(\mathsf{EA})$$

- (Beklemishev)
  - $\mathsf{EA}^+ + \mathsf{Rfn}_{\Sigma_2}(\mathsf{EA}) \equiv \mathsf{EA}^+ + \mathsf{I}\Pi_1^-$.
  - $\mathsf{EA}^+ + \Pi_1\text{–}\mathsf{IR} \equiv T_\omega$ (iterated consistency).

Proposition (Visser, CFL)

1. $\mathsf{EA} + \mathsf{Rfn}_{\Sigma_2}(\mathsf{EA}) \equiv \mathsf{EA} + \mathsf{I}\Pi_1^-$.
2. $\mathsf{EA} + \mathsf{Rfn}_{\Sigma_1}(\mathsf{EA}) \equiv [\mathsf{EA}, \Pi_1^-\text{–}IR_0]$.
3. $\mathsf{EA} + \Pi_1\text{–}IR \equiv T_\omega$ *(iterated consistency).*

Transfering the results to Π_1–induction

Let θ be a sentence.

- Assume $EA + Rfn_{\Sigma_2}(EA) \vdash_m \theta$. Then

$\theta \in \Pi_2$	$EA + Rfn_{\Pi_2}(EA) \vdash_m \theta$
$\theta \in \mathcal{B}(\Sigma_1)$	$EA + Rfn_{\mathcal{B}(\Sigma_1)}(EA) \vdash_m \theta$
$\theta \in \Pi_1$	$EA_m \vdash \theta$

- Assume $EA^+ + I\Pi_1^- \vdash_m \theta$. Then

$\theta \in \Pi_2$	$EA^+ + \mathbf{?} \vdash_m \theta$
$\theta \in \mathcal{B}(\Sigma_1)$	$EA^+ + \mathbf{?} \vdash_m \theta$
$\theta \in \Pi_1$	$[EA^+, \Pi_1\text{–IR}]_m \vdash \theta$ (*Beklemishev*)

Questions

Let T be an extension of $I\Delta_0$. Then

- Can we isolate induction principles $P1$ and $P2$ such that if $T + I\Pi_1^- \vdash_m \theta$, then

$\theta \in \Pi_2$	$T + \mathbf{P1} \vdash_m \theta$
$\theta \in \mathcal{B}(\Sigma_1)$	$T + \mathbf{P2} \vdash_m \theta$

- Can we prove that for each $\theta \in \Pi_1 \cap \text{Sent}$, if $T + I\Pi_1^- \vdash_m \theta$, then

$$[T, \Pi_1\text{–IR}]_m \vdash \theta \ ?$$

Local induction

- We denote by $I(\Gamma, \mathcal{K}_n)$ the following induction scheme

$$\varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow (U_\delta \rightarrow \forall x(\delta(x) \rightarrow \varphi(x)))$$

where $\varphi(x) \in \Gamma$, $\delta(x) \in \Sigma_n^-$ and U_δ is the sentence

$$\forall x_1\, \forall x_2\, (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2)$$

- $(\Gamma, \mathcal{K}_n)$–IR denotes the following inference rule:

$$\frac{\varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1))}{U_\delta \rightarrow \forall x(\delta(x) \rightarrow \varphi(x))}$$

where $\varphi(x) \in \Gamma$ and $\delta(x) \in \Sigma_n^-$.

- $I(\Gamma^-, \mathcal{K}_n)$ and $(\Gamma^-, \mathcal{K}_n)$–IR denote the parameter free versions
- The rule $(\Gamma^-, \mathcal{K}_n)$–$\mathrm{IR}_0$ is defined in a similar way.

Connection with parameter free Π_1–induction

- Over $I\Delta_0$, $I\Pi_1^- \equiv I(\Sigma_1^-, \mathcal{K}_1)$
  - The equivalence is one-to-one: one instance of the first scheme suffices to derive a given instance of the second one (and viceversa).
- For every theory T extending $I\Delta_0$,

$$[T, (\Sigma_1^-, \mathcal{K}_1)\text{–IR}_0] \equiv [T, \Pi_1^-\text{–IR}_0]$$

  - It is again a "one–to–one equivalence".

The main result

Let T be $I\Delta_0 + \forall x \exists y \varphi(x,y)$, where $\varphi(x,y) \in \Delta_0$ and $I\Delta_0$ proves that $\varphi(x,y)$ defines a nondecreasing function. Let $m \geq 1$ and let θ be a sentence.

- Assume $T + I(\Sigma_1^-, \mathcal{K}_1) \vdash_m \theta$. Then

$\theta \in \Pi_2$	$[T, (\Sigma_1, \mathcal{K}_1)\text{–IR}] \vdash_m \theta$
$\theta \in \mathcal{B}(\Sigma_1)$	$[T, (\mathcal{B}(\Sigma_1)^-, \mathcal{K}_1)\text{–IR}] \vdash_m \theta$

- If $\theta \in \mathcal{B}(\Sigma_1)$ and $T + I\Pi_1^- \vdash_m \theta$, then there exist sentences $\pi_1, \ldots, \pi_r \in \Pi_1$ and $\sigma_1, \ldots, \sigma_r \in \Sigma_1$ such that $\vdash \bigvee_{j=1}^r (\sigma_j \wedge \pi_j)$ and for each $j = 1, \ldots, r$,

$$[T + \sigma_j \wedge \pi_j, \Pi_1^-\text{–IR}_0] \vdash_m \theta$$

- If in addition $\theta \in \Pi_1$, then

$$[T + \sigma_j \wedge \pi_j, \Pi_1\text{–IR}]_m \vdash \theta$$

Some ideas from the proof

- Two key points:
  - Adamowicz–Bigorajska–Kaye–Mints–Ratajczyk's Theorem has a local version.
  - A local version of the equivalence between applications of Σ_1–IR and iteration holds.
- For every $m \geq 1$ and $\theta \in \Pi_2$

$$\text{If } T + I(\Sigma_1^-, \mathcal{K}_1) \vdash_m \theta \text{ then } [T, (\Sigma_1, \mathcal{K}_1)\text{–IR}]_m \vdash \theta$$

- **(Local iteration theorem)** The following theories are equivalent:
  - $T + (\Sigma_1, \mathcal{K}_1)$–IR.
  - $[T, (\Sigma_1, \mathcal{K}_1)\text{–IR}]$.
  - $T + \forall u \in \mathcal{K}_1 \, \forall x \, \exists y \, (f^u(x) = y)$.
    (where $f(x) = (x+1)^2 + (\mu x)\varphi(x, y)$).

Parameter free Π_2–induction

In the case $n = 2$, we have:

- $I\Pi_2^- \equiv I(\Sigma_2^-, \mathcal{K}_2)$.
- $I(\Sigma_2, \mathcal{K}_2)$ is Π_3–conservative over $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR.
- $I\Sigma_1$ extends $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR.
  - Reduction: $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)\text{–IR} \equiv I\Sigma_1^- + (I\Delta_0 + (\Sigma_2, \mathcal{K}_2)\text{–IR})$.
  - A refinement of the (proof of) Local Iteration Theorem shows that $I\Sigma_1$ extends $I\Delta_0 + (\Sigma_2, \mathcal{K}_2)$–IR.
  - It follows that $I\Pi_2^-$ is Π_3–conservative over $I\Sigma_1$.
- **Question**: Let $\theta \in \Pi_3 \cap \text{Sent}$ such that $I\Sigma_1^- + I(\Sigma_2^-, \mathcal{K}_2) \vdash_m \theta$.
  - Does $[I\Sigma_1^-, (\Sigma_2, \mathcal{K}_2)\text{–IR}] \vdash_m \theta$ hold?
- Assume that $I\Delta_0 + I\Pi_2^- \vdash_m \theta$ with $\theta \in \mathcal{B}(\Sigma_2) \cap \text{Sent}$ or $\theta \in \Pi_2 \cap \text{Sent}$.

What can we say in these cases?