# SUBMODEL LATTICES OF NERODE SEMIRINGS

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Let $\mathbb{N} = (\omega, +, \times, 0, 1, \leq)$ be the standard model of arithmetic, and then True Arithmetic is $\mathsf{TA} = \mathrm{Th}(\mathbb{N})$. For $n < \omega$, let $\mathsf{TA}_n = \mathsf{TA} \cap \Pi_n$. If $\mathcal{N} \models \mathsf{TA}_2$, then let $\mathcal{L}(\mathcal{N}) = \{\mathcal{M} \subseteq \mathcal{N} : \mathcal{M} \models \mathsf{TA}_2\}$. Then $\mathcal{L}(\mathcal{N})$ is a complete lattice, so it makes sense to refer to those $\mathcal{N} \models \mathsf{TA}_2$ that are finitely generated, and these are by definition the **Nerode semirings**. I will be discussing the following question: *For which lattices $L$ is there a Nerode semiring $\mathcal{N}$ such that $\mathcal{L}(\mathcal{N}) \cong L$?* There is a corresponding question for models of $\mathsf{PA}$ and the lattice of elementary substructures. We will see that there are some similarities and some differences between the Nerode semiring question and the $\mathsf{PA}$ question. (This is joint work with Volodya Shavrukov.)