Submodel Lattices of Nerode Semirings

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The standard model of Arithmetic is

$$\mathbb{N} = (\omega, +, \times, 0, 1, \leq).$$

$$\mathsf{TA} = \mathsf{Th}(\mathbb{N}) = \text{True Arithmetic}$$

$$\mathsf{TA}_n = \mathsf{TA} \cap \Pi_n$$

Our concern: TA_2 and its models.

Two comments about TA_2:

1. *Matijasavic's Theorem* $\Longrightarrow$
$$\mathrm{TA}_2 = \mathrm{TA} \cap \forall\exists.$$
2. If $\mathcal{N}$ is a model of TA_2 and $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} \models \mathrm{TA}_2$ iff M closed under all computable functions.

If $\mathcal{N} \models \mathrm{TA}_2$, then

$$\mathcal{L}(\mathcal{N}) = \{\mathcal{M} \subseteq \mathcal{N} : \mathcal{M} \models \mathrm{TA}_2\}.$$

Hence, $\mathcal{L}(\mathcal{N})$ is a complete lattice.

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Definition/Theorem: *$\mathcal{N}$ is a Nerode semiring iff $\mathcal{N}$ is finitely generated.*

Tom McLaughlin, 2010:
"Some observations on the substructure lattice of a Δ_1-ultrapower"

Δ_1-ultrapower $\leftrightsquigarrow$ Nerode semiring.

What are the possible lattices $\mathcal{L}(\mathcal{N})$ for $\mathcal{N}$ a Nerode semiring?

If $\mathcal{N} \models \mathrm{PA}$, then

$$\mathrm{Lt}(\mathcal{N}) = \{\mathcal{M} \subseteq \mathcal{N} : \mathcal{M} \preccurlyeq \mathcal{N}\}.$$

If $\mathcal{N} \models \mathrm{TA}_2$, then

$$\mathcal{L}(\mathcal{N}) = \{\mathcal{M} \subseteq \mathcal{N} : \mathcal{M} \models \mathrm{TA}_2\}.$$

What are the possible finite lattices $\mathcal{L}(\mathcal{N})$ for $\mathcal{N}$ a Nerode semiring?

Theorem: *If L is a finite lattice, then the following are equivalent*:

(1) *There is a Nerode semiring $\mathcal{N}$ such that $\mathcal{L}(\mathcal{N}) \cong L$.*

(2) *There is L_1 such that $L \cong \mathbf{2} \oplus L_1$ and L_1 has n-CPP representations for all $n < \omega$.*

What's all this mean?

CPP = Canonical Partition Property

Lots of finite lattices have n-CPP representations for all $n < \omega$. All distributive lattices do and many others are known to have. Possibly, every finite lattice does.

Theorem: *If L is a finite lattice, then the following are equivalent*:

(1) *There is a Nerode semiring* $\mathcal{N}$ *such that* $\mathcal{L}(\mathcal{N}) \cong L$.
(2) *There is* L_1 *such that* $L \cong \mathbf{2} \oplus L_1$ *and* L_1 *has* $\mathbf{n}$*-CPP representations for all* $\mathbf{n} < \omega$.

If L is a finite lattice, $L \cong \mathbf{2} \oplus L_1$ and L_1 has n-CPP representations for all $n < \omega$, then there is $\mathcal{N} \models PA$ such that $\mathrm{Lt}(\mathcal{N}) \cong L$.

In fact, we can get $\mathcal{N} \models \mathrm{TA}$.

These are not the only finite L - for example, any distributive L works.

Theorem: *Suppose that* $\mathcal{N} \models TA_2$.

- *If* $\mathcal{N}$ *has at least* 2 *nonstandard skies, then*
  $$0'' \in SSy(\mathcal{N}).$$
- *If* $0'' \in SSy(\mathcal{N})$, *then*
  - *the skies of* $\mathcal{N}$ *are densely ordered*;
  - *for all possible finite* L, *there is* $\mathcal{M} \in \mathcal{L}(\mathcal{N})$ *such that* $\mathcal{L}(\mathcal{M}) \cong L$;
  - $\mathcal{N}$ *has infinitely many atoms.*

Lemma: *If* $\mathcal{N} \models TA_2$ *and* $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{N}$ *are Nerode semirings cofinal in* $\mathcal{N}$, *then* $\mathcal{M}_1 \cap \mathcal{M}_2$ *is cofinal in* $\mathcal{N}$.

What are the possible chains $\mathcal{L}(\mathcal{N})$ for $\mathcal{N}$ a Nerode semiring?

Theorem: *If L is a (possibly infinite) chain, then the following are equivalent*:

(1) *There is a Nerode semiring $\mathcal{N}$ such that $\mathcal{L}(\mathcal{N}) \cong L$.*

(2) *L is complete, has a coatom and has at most countably many immediate successors, and every element is the sup of those immediate successors below it.*

Definition: $\mathcal{M} \models \text{TA}_2$ is existentially closed (e.c.) if whenever $\mathcal{M} \subseteq \mathcal{N} \models \text{TA}_2$, then $\mathcal{M} \prec_1 \mathcal{N}$.

Joram Hirschfeld & William H. Wheeler, 1975:
"Forcing, arithmetic, division rings." (especially Part 2)

Are there e.c. models of TA_2? Yes

Are there e.c. Nerode semirings? Yes

Σ_1-ultrapower $\leftrightsquigarrow$ e.c. Nerode semiring.

What are the possible lattices $\mathcal{L}(\mathcal{N})$ for $\mathcal{N}$ an e.c. Nerode semiring?

Theorem: *If $\mathcal{N}$ is an e.c. Nerode semiring, then $\mathcal{N}$ has a single nonstandard sky* (i.e. $0'' \notin SSy(\mathcal{N})$).

Theorem: *If L is a chain and there is a Nerode semiring $\mathcal{N}$ such that $\mathcal{L}(\mathcal{N}) \cong L$, then there is an e.c. Nerode semiring $\mathcal{N}$ such that $\mathcal{L}(\mathcal{N}) \cong L$.*

Theorem: *There is an e.c. Nerode semiring such that $\mathcal{L}(\mathcal{N}) \cong \mathbf{2} \oplus \mathbf{B}_3$.*

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