Understanding $B\Sigma_1 + \exp$ via WKL_0^*

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This talk

Conservation Theorem (SSDP)

The following are equivalent for $M \models \mathsf{PA}^-$.

(a) $M \models \mathsf{B}\Sigma_1 + \exp$.

(b) M expands to $(M, \mathscr{X}) \models \mathsf{WKL}_0^*$.

Plan

1. Introduction
2. Arithmetized-Completeness-Theorem constructions
3. Arithmetized-Completeness-Theorem constructions iterated
4. Conclusion

$\mathrm{B}\Sigma_1 + \exp$ and WKL_0^*

Conservation Theorem (SSDP)

The following are equivalent for $M \models \mathrm{PA}^-$.

(a) $M \models \mathrm{B}\Sigma_1 + \exp$.

(b) M expands to $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$.

$\mathrm{B}\Sigma_1$ consists of PA^- and

- Δ_0-induction ($\mathrm{I}\Delta_0$);
- Σ_1-collection.

1st order arithmetic

WKL_0^* consists of extensionality, PA^-, and

- Σ_0^0-induction ($\mathrm{I}\Sigma_0^0$);
- totality of exponentiation (exp);
- Δ_1^0-comprehension (Δ_1^0-CA); and
- Weak König Lemma (WKL), i.e., every unbounded *binary* tree has an unbounded path.

(The first three items together form RCA_0^*.)

2nd order arithmetic

The Conservation Theorem

Conservation Theorem (SSDP)

The following are equivalent for $M \models \mathsf{PA}^-$.

(a) $M \models \mathsf{B}\Sigma_1 + \exp$.

(b) M expands to $(M, \mathscr{X}) \models \mathsf{WKL}_0^*$.

Proof

(Simpson–Smith 1986) $\mathsf{WKL}_0^* \vdash \mathsf{B}\Sigma_1 + \exp$.

(Folklore) If $M \models \mathsf{I}\Delta_0 + \exp$ has a proper end extension $K \models \mathsf{I}\Delta_0$, then $(M, \mathrm{SSy}_M(K)) \models \mathsf{WKL}_0^*$.

(Dimitracopoulos–Paschalis) Every $M \models \mathsf{B}\Sigma_1 + \exp$ has a proper end extension $K \models \mathsf{I}\Delta_0$. □

Remark

Simpson and Smith proved this for *countable* M using forcing.

Arithmetized Completeness Theorem

Theorem (Simpson)

The following are equivalent over RCA_0^*.

(a) WKL.

(b) Gödel's Completeness Theorem, i.e., every consistent theory in first-order logic has a model.

Arithmetized Completeness Theorem for WKL_0^*

Let $(M, \mathscr{X}) \models \mathsf{WKL}_0^*$ and $T \supseteq \mathsf{PA}^-$ be a theory coded in $(M, \mathscr{X})$. If $(M, \mathscr{X}) \models \mathrm{Con}(T)$, then there is $K \models T$ end extending M.

Theorem (Dimitracopoulos–Paschalis)

Every $M \models \mathsf{B}\Sigma_1 + \exp$ has a proper end extension $K \models \mathsf{I}\Delta_0$.

Theorem (Wilkie–Paris 1987)

$\mathsf{B}\Sigma_1 + \exp \vdash \mathrm{TabCon}(\mathsf{I}\Delta_0^*)$.

Cuts in models of Peano

Π_1-overspill at $\mathbb{N}$

Theorem (Kirby–Paris 1978, Enayat–W)

Let $M \models \mathrm{I}\Delta_0 + \exp$ that is short Π_1-recursively saturated.
Then the following are equivalent.

(a) $M \models \mathrm{B}\Sigma_1 + \Pi_1\text{-}\mathrm{Th}(\mathrm{PA})$.

(b) M has a proper end extension $K \models \mathrm{PA}$.

Proof (essentially in Mc Aloon 1978)

- Expand M to $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$.
- Reflection implies $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{I}\Sigma_n)$ for every $n \in \mathbb{N}$.
- So $M \models \mathrm{Con}(\mathrm{I}\Sigma_n)$ for every $n \in \mathbb{N}$.
- Overspill implies $M \models \mathrm{Con}(\mathrm{I}\Sigma_\nu)$ for some $\nu > \mathbb{N}$.
- There is $K \models \mathrm{I}\Sigma_\nu \supseteq \mathrm{PA}$ end extending M. □

Coded sets

Theorem (Dimitracopoulos–Paschalis), rephrased

The following are equivalent for $M \models \mathrm{I}\Delta_0 + \exp$.

(a) $M \models \mathrm{B}\Sigma_1$.

(b) M has a proper end extension $K \models \mathrm{I}\Delta_0$.

Theorem (Enayat–W)

The following are equivalent for a countable $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$.

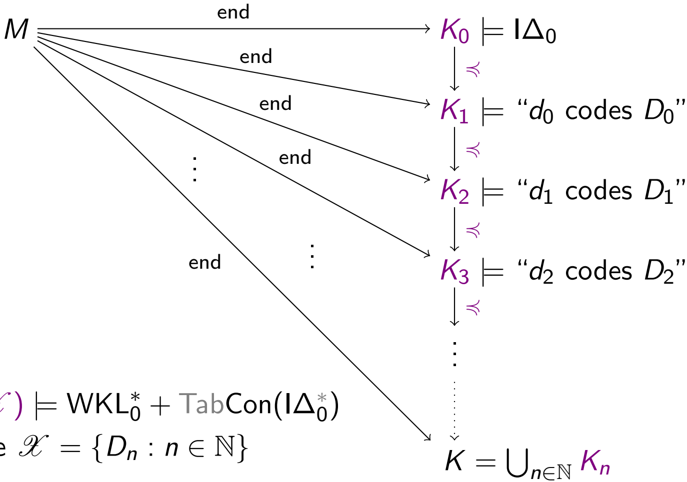
(a) $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$.

(b) M has a proper end extension $K \models \mathrm{I}\Delta_0$ where $\mathrm{SSy}_M(K) = \mathscr{X}$.

Proof

A standard overspill argument shows (b) $\Rightarrow$ (a).

End extending M to $K \models \mathrm{I}\Delta_0$ with $\mathrm{SSy}_M(K) = \mathscr{X}$



$(M, \mathscr{X}) \models \mathrm{WKL}_0^* + \mathrm{TabCon}(\mathrm{I}\Delta_0^*)$
where $\mathscr{X} = \{D_n : n \in \mathbb{N}\}$

$$(M, \mathscr{X}) \models \mathrm{TabCon}(\mathrm{ElemDiag}(K_n) + \text{"}d_n \text{ codes } D_n\text{"})$$

□

Completions of PA

Theorem (Kirby–Paris 1978, Enayat–W)

Let $M \models \mathrm{I}\Delta_0 + \exp$ that is short Π_1-recursively saturated.
Then the following are equivalent.

(a) $M \models \mathrm{B}\Sigma_1 + \Pi_1\text{-}\,\mathrm{Th}(\mathrm{PA})$.

(b) M has a proper end extension $K \models \mathrm{PA}$.

Theorem (Wilkie 1977, Enayat–W)

Let $M \models \mathrm{I}\Delta_0 + \exp$ that is short Π_1-recursively saturated.
The following are equivalent for a complete consistent $T \supseteq \mathrm{PA}$.

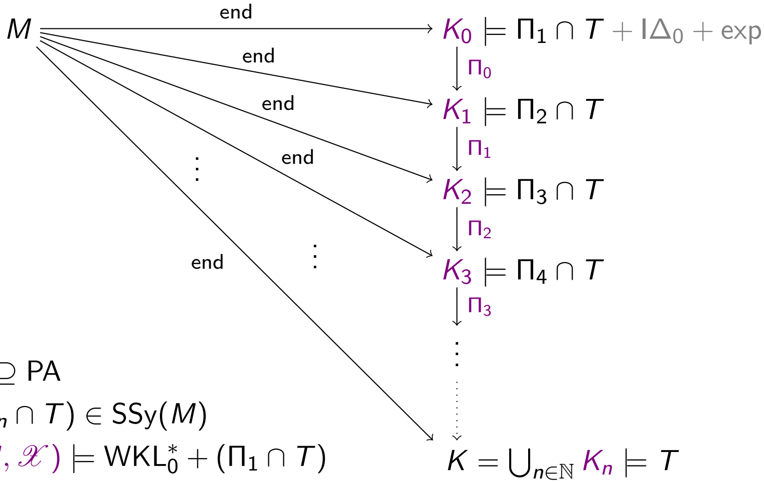
(a) $M \models \mathrm{B}\Sigma_1 + (\Pi_1 \cap T)$ and $(\Pi_n \cap T) \in \mathrm{SSy}(M)$ for all $n \in \mathbb{N}$.

(b) M has a proper end extension $K \models T$.

Proof

(b) $\Rightarrow$ (a) is a simple application of the Sat_{Π_n}'s.

End extending M to $K \models T$ following Lessan and Schmerl



$T \supseteq \mathrm{PA}$

$(\Pi_n \cap T) \in \mathrm{SSy}(M)$

$(M, \mathscr{X}) \models \mathrm{WKL}_0^* + (\Pi_1 \cap T)$

$$M \subseteq_{\mathrm{e}} K_n \models \mathrm{Con}\big(\Pi_n\text{-}\mathrm{Diag}(\cdot) + (\Pi_{n+1} \cap T)\big) \in \Pi_{n+1} \cap T \quad \square$$

Further applications

satisfied if $M \models \mathrm{I}\Sigma_1$

Theorem (Enayat–W)

Let $M \models \mathrm{B}\Sigma_1 + \exp$ be nonstandard and short $(\Sigma_1 \cup \Pi_1)$-rec. sat.
The following are equivalent for a complete consistent $T \supseteq \mathrm{PA}$.

(a) $M \models \Pi_2 \cap T$ and $(\Pi_n \cap T) \in \mathrm{SSy}(M)$ for all $n \in \mathbb{N}$.

(b) M has arbitrarily large initial segments $I \models T$.

Theorem (Schmerl 2014)

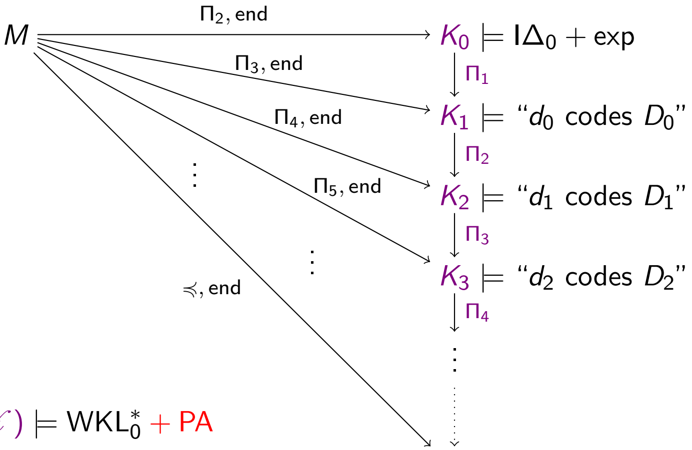
The following are equivalent for a countable $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$.

(a) $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$ and $\mathscr{X} \supseteq \mathrm{Def}(M)$.

(b) $\mathscr{X} = \mathrm{SSy}_M(K)$ for some $K \supsetneq_{\mathrm{e}} M$.

parametrically definable sets

Finding $K \succcurlyeq_{\mathrm{e}} M$ with $\mathrm{SSy}_M(K) = \mathscr{X}$ following Kaufmann



$(M, \mathscr{X}) \models \mathrm{WKL}_0^* + \mathrm{PA}$

with $\mathscr{X} = \{D_n : n \in \mathbb{N}\} \supseteq \mathrm{Def}(M)$

$K = \bigcup_{n \in \mathbb{N}} K_n$

$$K_n \succcurlyeq_{\Pi_{n+2}} M \models \mathrm{Con}\big(\Pi_{n+1}\text{-}\mathrm{Diag}(\cdot) + \Pi_{n+3}\text{-}\mathrm{Diag}(M)\big) \in \Pi_{n+2}$$

□

Summary

Conservation Theorem (SSDP)

The following are equivalent for $M \models \mathrm{PA}^-$.

(a) $M \models \mathrm{B}\Sigma_1 + \exp$.

(b) M expands to $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$.

is useful for

(1) iterating Arithmetized-Completeness-Theorem constructions;

(2) controlling subsets coded in end extensions.

Questions

(i) Can we still pass on to second-order arithmetic without exp?

(ii) How necessary is PA in our theorems/arguments?