Use of nonstandard models in Reverse Mathematics

Keita Yokoyama

Japan Advanced Institute of Science and Technology

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What is Reverse Mathematics?

Hilbert's reductionism program (1920s):

Find a good axiomatic system T for the entire mathematics, and prove the 'consistency of T' by a 'finitistic method'.

- This program failed because of Gödel's incompleteness theorem (1930).

$\Rightarrow$ Which axioms are exactly needed for mathematics?

$\Rightarrow$ **Reverse Mathematics**

H. Friedman's theme (1976):

very often, if a theorem τ of ordinary mathematics is proved from the "right" axioms, then τ is equivalent to those axioms over some weaker system in which itself is not provable.

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Reverse Mathematics program (Friedman Simpson program)

1. Formalize the theorem τ of "core of math" within an appropriate axiomatic system.
2. Find the weakest axiom T in which we can prove τ.
3. Classify "core of math" using the logical strength.

   ("core of math": basic theorems of analysis, algebra, geometry, etc.)

Study the strength of various theorems by this method.

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language of second order arithmetic ($\mathcal{L}_2$)

Definition (language of second order arithmetic)

number variables: $x, y, z, \ldots$ set variables: $X, Y, Z, \ldots$
constant symbols: $0, 1$ function symbols: $+, \cdot$
relation symbols: $=, <, \in$

Classes of formulas

bounded formula: all quantifiers are of the form $\forall x < y$, $\exists x < y$.

arithmetical formulas:(θ: bounded formula)
Σ^0_n formula: $\exists x_1 \forall x_2 \dots x_n \theta$
Π^0_n formula: $\forall x_1 \exists x_2 \dots x_n \theta$

analytic formula:(φ: arithmetical formula)
Σ^1_n formula:$\exists X_1 \forall X_2 \dots X_n \varphi$
Π^1_n formula:$\forall X_1 \exists X_2 \dots X_n \varphi$

Induction axioms

- Σ^i_j induction ($\mathrm{I}\Sigma^i_j$): for any $\varphi(x) \in \Sigma^i_j$,

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x).$$

- Δ^i_j induction ($\mathrm{I}\Delta^i_j$): for any $\varphi(x) \in \Sigma^i_j$ and $\psi(x) \in \Pi^i_j$,

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \to (\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)).$$

- Σ^i_j bounding ($\mathrm{B}\Sigma^i_j$): for any $\varphi(x, y) \in \Sigma^i_j$,

$$\forall x < u \exists y \varphi(x, y) \to \exists v \forall x < u \exists y < v \varphi(x, y).$$

Note that $\mathrm{B}\Sigma^0_{j+1} = \mathrm{I}\Delta^0_j$ over $\mathrm{I}\Sigma^0_1$. (Slaman 2004)

Comprehension axioms

- Σ^i_j (Π^i_j) comprehension: for any $\varphi(x) \in \Sigma^i_j$,

$$\exists X \forall x(\varphi(x) \leftrightarrow x \in X).$$

- Δ^i_j comprehension: for any $\varphi(x) \in \Sigma^i_j$ and $\psi(x) \in \Pi^i_j$,

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \exists X \forall x(\varphi(x) \leftrightarrow x \in X).$$

- weak König's lemma:
  for any infinite tree $T \subseteq 2^{<\mathbb{N}}$, $\exists X \forall n(X[n] \in T)$,
  where $X[n] = \langle X(0), \ldots, X(n-1)\rangle$.

Subsystems of second-order arithmetic

Big five plus one

- RCA_0: "discrete ordered semi-ring"+Σ^0_1 induction +Δ^0_1 comprehension.
- WWKL_0: RCA_0 + weak weak König's lemma.
- WKL_0: RCA_0 + weak König's lemma.
- ACA_0: RCA_0 + Σ^1_0 comprehension.
- ATR_0: RCA_0 + arithmetical transfinite recursion.
- $\Pi^1_1\mathrm{CA}_0$: RCA_0 + Π^1_1 comprehension.

Subsystems of second-order arithmetic

Big five plus one

- RCA_0: In this system, we need to prove everything "recursively".
- WWKL_0: We can use the notion of measure for closed set.
- WKL_0: We can use Σ^0_1-separation, or we can use Hiene/Borel compactness.
- ACA_0: We can use number quantifier freely, or we can use sequential compactness.
- ATR_0: We can compare well orderings.
- $\Pi^1_1\mathsf{CA}_0$: We can check well-foundedness.

Reverse Mathematics

Theorem

The following are provable within RCA_0.

1. *The structure theorem for finitely generated abelian group.*
2. *Mean value theorem.*
3. *Implicit function theorem.*
4. *Taylor's expansion theorem for holomorphic function.*
5. *Baire Category theorem.*
6. *The Riemann mapping theorem for a polygonal region.*
7. ...

Reverse Mathematics

Theorem

The following are equivalent over RCA_0*.*

1. WKL_0.
2. *Hiene Borel compactness for* $[0,1]$*.*
3. *Completeness theorem/ compactness theorem.*
4. *Uniqueness of algebraic closures of a countable field.*
5. *Every continuous function on* $[0,1]$ *has a maximum.*
6. *The Jordan–Schönflies theorem.*
7. *The Cauchy integral theorem.*
8. *The Riemann mapping theorem for a Jordan region.*
9. ...

Reverse Mathematics

Theorem

The following are equivalent over RCA_0.

1. ACA_0.
2. *Ramsey's theorem:* RT^n *for* $n \geq 3$.
3. *Every countable countable vector space has a basis.*
4. *Every countable commutative ring has a maximal ideal.*
5. *Arzela/Ascoli's theorem.*
6. *The Riemann mapping theorem (over* WKL_0*).*
7. . . .

We will check the strength of various theorems in this way.

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RM in a weaker base system

Sometimes, we weaken the base system.

Review (Big five)

- RCA_0: *basic axioms: "discrete ordered semi-ring" + Σ^0_1 induction + recursive comprehension.*
- WKL_0: RCA_0 + *weak König's lemma.*
- ACA_0: RCA_0 + *arithmetical comprehension.*
- ATR_0: RCA_0 + *arithmetical transfinite recursion.*
- $\Pi^1_1\mathrm{CA}_0$: RCA_0 + Π^1_1*-comprehension.*

RM in a weaker base system

Sometimes, we weaken the base system.

Definition

- RCA_0^*: basic axioms: "discrete ordered semi-ring" + "for any x, 2^x exists" + Σ^0_0-induction + recursive comprehension.
- WKL_0^*: RCA_0^* + weak König's lemma.
- RCA_0: RCA_0^* + Σ^0_1-induction.
- WKL_0: RCA_0 + weak König's lemma.
- ACA_0: RCA_0^* + arithmetical comprehension.
- ATR_0: RCA_0^* + arithmetical transfinite recursion.
- $\Pi^1_1\mathsf{CA}_0$: RCA_0^* + Π^1_1-comprehension.

Reverse mathematics over RCA_0

(Over RCA_0)

The following are provable within RCA_0.

- Every finitely generated vector space has a basis.
- For every countable field K, every polynomial $f(x) \in K[x]$ has only finitely many roots in K.

The following are equivalent to WKL_0.

- Every countable ring has a prime ideal.
- Σ^0_1-determinacy in Cantor space.
- Every countable Peano system is isomorphic to $(\mathbb{N}, 0. + 1)$.

The following are equivalent to ACA_0.

- Every countable ring has a maximal ideal.
- Ramsey's theorem RT^3_2.

Reverse mathematics over RCA_0^*

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The following are equivalent to ACA_0.

- Every countable ring has a maximal ideal.
- ~~Ramsey's theorem RT^3_2.~~ $\Leftarrow \mathrm{I}\Sigma^0_1$ is needed!!

Outline

1. Computability vs NS models –on Ramsey's thm–
   - Formalizing Computability
   - Hybrid method Computability and NS models
   - Classical methods survive
2. Nonstandard analysis and RM
3. Some recent ideas
   - Random preserving extension
   - RM over RCA_0^*

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Big five plus one vs Computability

Big five plus one

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Big five plus one vs Computability

Big five plus one

- RCA_0: Turing reducibility.
- $WWKL_0$: Martin-Löf random real.
- WKL_0: Low basis theorem for Π^0_1-classes.
- ACA_0: Turing jump.
- ATR_0: (Hyper arithmetical reducibility).
- $\Pi^1_1CA_0$: Hyper jump.

Ramsey's theorem

Question

What is the strength of (several versions of) Ramsey's theorem?

Definition (Ramsey's theorem.)

- RT^n_k: for any $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
- $\mathrm{RT}^n_\infty := \forall k\ \mathrm{RT}^n_k$.
- $\mathrm{RT}^\infty_\infty := \forall n\ \mathrm{RT}^n$.

(We often omit ∞.)

Many results are derived from computability theory (= results on ω-models).

What is the strength of Ramsey's theorem?

Proposition

ACA_0 *proves* $\forall n \forall k (\mathrm{RT}^n_k \to \mathrm{RT}^{n+1}_k)$.

Proof.

The usual proof works within ACA_0. □

Theorem (Jockusch 1972)

Over RCA_0, RT^3_2 *implies* ACA_0.

Proof.

There exists a computable coloring for $[\mathbb{N}]^3$ whose homogeneous set always computes $\mathbf{0}'$. □

Thus, for $n \geq 3$, $\mathrm{RT}^n_2 = \mathrm{RT}^n = \mathrm{ACA}_0$.

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Separation

Using computability theory, we have the following.

- $\mathsf{RCA}_0 \nvdash \mathrm{RT}^2_2$. (Specker 1971)
  $\Uparrow$ there exists a computable coloring
  which has no computable homogeneous set.
  Later, $\mathsf{RCA}_0 + \mathrm{RT}^2_2 \vdash \mathrm{DNR}$ (HJHLS 2008).
- $\mathsf{RCA}_0 + \mathrm{RT}^2 \nvdash \mathrm{RT}^3_2$. (Seetapun 1995)
  $\Uparrow$ Cone avoidance for coloring for pairs.
  Later, low$_2$-basis theorem (CJS 2001).
- $\mathsf{RCA}_0 + \mathrm{RT}^2 \nvdash \mathsf{WKL}_0$. (Liu 2011)
  $\Uparrow$ DNR_2 avoidance for coloring for pairs.

Combining with the first-order strength, we have,

$$\mathrm{RT}^1_2 < \mathrm{RT}^1 < \mathrm{RT}^2_2 < \mathrm{RT}^2 < \mathrm{RT}^3_2 = \mathrm{RT}^3 = \cdots = \mathrm{RT}^n < \mathrm{RT}.$$

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Ramsey's theorem

Question

What is the strength of RT^2_2?

Formalizing low$_2$ basis theorem to the following.

Theorem (Cholak/Jockusch/Slaman)

For any $(M, S) \models \mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$ and for any coloring $P : [M]^2 \to 2$ in S, there exists $H \subseteq M$ such that H is a homogeneous set for P and $(M, S \cup \{G\}) \models \mathrm{I}\Sigma^0_2$.

Theorem (Cholak/Jockusch/Slaman)

$\mathrm{WKL}_0 + \mathrm{RT}^2_2 + \mathrm{I}\Sigma^0_2$ is a Π^1_1-conservative extension of $\mathrm{I}\Sigma^0_2$.

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Recent hybrid method

RT^2_2 can be decomposed into computable notions as follows:

- $\mathrm{RT}^2_2 = \mathrm{D}^2_2 + \mathrm{COH}$.
  - D^2_2: any Δ^X_2 set contains an infinite set or is disjoint from an infinite set.
  - COH: any sequence of sets $\langle R_i \mid i \in \mathbb{N}\rangle$ has a cohesive set.

Whether D^2_2 is equivalent to RT^2_2 or not was a long term open question.

Theorem (Chong/Slaman/Yang)

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Idea.

- (Jockusch) There exists a computable coloring P such that any homogeneous set for P is not low.
- (Downey/Hirschfeldt/Lempp/Solomon) There exists a Δ_2-set which contains/is disjoint from no infinite low set.

Lemma (not enough for the theorem)

There exists a model $M \models \mathrm{B}\Sigma_2 + \neg\mathrm{I}\Sigma_2$ such that any $\Delta_2(M)$-set contains an infinite low set or is disjoint from infinite low set.

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Indicator approach

Classical argument is still useful.
We will see several consequences of Paris's indicator argument using the "density for finite colorings".

Definition (finite coloring)

- (n,k)-finite coloring is a function $P : [F]^n \to k$ where $F = \mathrm{dom}(P) \subseteq_{\mathrm{fin}} \mathbb{N}$.
- (n,∞)-finite coloring is a function $P : [F]^n \to k$ where $F = \mathrm{dom}(P) \subseteq_{\mathrm{fin}} \mathbb{N}$ and $k \leq \min F$.
- (∞,∞)-finite coloring is a function $P : [F]^n \to k$ where $F = \mathrm{dom}(P) \subseteq_{\mathrm{fin}} \mathbb{N}$ and $n, k \leq \min F$.

Density notion

Let $\alpha, \beta \in \omega \cup \{\infty\}$.

Definition (RCA_0)

- A finite set X is said to be 0-*dense*(α, β) if $|X| > \min X$ (relatively large).
- A finite set X is said to be $m+1$-*dense*(α, β) if for any (α, β)-finite coloring P with $\mathrm{dom}(P) = X$, there exists $Y \subseteq X$ which is m-dense(α, β) and P-homogeneous.

Note that "X is m-dense(α, β)" can be expressed by a Σ^0_0-formula.

Paris-Harrington principle

Definition

- $m\mathrm{PH}^{\alpha}_{\beta}$: for any $a \in \mathbb{N}$ there exists an m-dense(α,β) set X such that $\min X > a$.
- $m\widetilde{\mathrm{PH}}^{\alpha}_{\beta}$:for any $X_0 \subseteq_{\mathrm{inf}} \mathbb{N}$, there exists an m-dense(α,β) set X such that $X \subseteq_{\mathrm{fin}} X_0$.

We write $\mathrm{ItPH}^{\alpha}_{\beta}$ for $\forall m\, m\mathrm{PH}^{\alpha}_{\beta}$.

- Original Paris's independent statement from PA is ItPH^{3}_{2}.
- Original Paris-Harrington principle is $1\mathrm{PH}^{\infty}_{\infty}$.
- They are both equivalent to the Σ_1-soundness of PA.

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Paris's argument

We fix $\alpha, \beta \in \omega \cup \{\infty\}$ such that $\alpha, \beta \geq 2$, or $\alpha = 1$ and $\beta = \infty$.

Lemma

If (M, S) is a countable model of RCA_0 and $X \subset M$ ($X \in S$ and M-finite) is m-dense(α, β) for some $m \in M \setminus \omega$, then there exists a cut $I \subseteq_e M$ such that $I \cap X$ is unbounded in I and $(I, S \upharpoonright I) \models \mathrm{WKL}_0 + \mathrm{RT}^{\alpha}_{\beta}$. Here, $S \upharpoonright I = \{I \cap X \mid X \in S\}$.

This lemma means that m-dense(α, β) defines an indicator function for $\mathrm{WKL}_0 + \mathrm{RT}^{\alpha}_{\beta}$.

Paris's argument

Let $\tilde{\Pi}^0_3$ be a class of formulas of the form $\forall X \varphi(X)$ where $\varphi \in \Pi^0_3$.

Theorem (essentially due to Paris)

$\mathrm{WKL}_0 + \mathrm{RT}^\alpha_\beta$ *is a conservative extension of* $\mathrm{RCA}_0 + \{m\widetilde{\mathrm{PH}}^\alpha_\beta \mid m \in \omega\}$ *with respect to* $\tilde{\Pi}^0_3$*-sentences.*

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$\mathrm{ItPH}^\alpha_\beta$ *is not provable from* $\mathrm{WKL}_0 + \mathrm{RT}^\alpha_\beta$.

In fact, we can strengthen this result to the following.

Theorem

Over $\mathrm{I}\Sigma_1$, $\mathrm{ItPH}^\alpha_\beta$ *is equivalent to the* Σ_1*-soundness of* $\mathrm{WKL}_0 + \mathrm{RT}^\alpha_\beta$.

Paris's argument

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Corollary

1. *The* $\tilde{\Pi}^0_3$*-part of* $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *is* $\mathrm{I}\Sigma^0_1 + \{m\widetilde{\mathrm{PH}}^2_2 \mid m \in \omega\}$*.*
2. *The* $\tilde{\Pi}^0_3$*-part of* $\mathsf{WKL}_0 + \mathrm{RT}^2_\infty$ *is* $\mathrm{I}\Sigma^0_1 + \{m\widetilde{\mathrm{PH}}^2_\infty \mid m \in \omega\}$*.*
3. $\mathrm{ItPH}^\infty_\infty$ *is not provable from* $\mathsf{ACA}_0 + \mathrm{RT}$*.*

Define GPH (generalized Paris-Harrington principle) as

> "every arithmetically definable infinite set contains *m*-dense(∞, ∞) set for any *m*".

Then, we have the following.

Theorem

$\mathrm{I}\Sigma_1 + \mathrm{GPH}$ *is the first-order part of* ACA_0'*, or equivalently* $\mathsf{ACA}_0 + \mathrm{RT}$*.*

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Density notion for WKL_0^*

Using the following weaker density notion, we know the strength of Ramsey's theorem over RCA_0^*.

Definition (within $\mathrm{B}\Sigma_1^0 + \exp$)

Let X be a finite set. Then,

- X is 0-dense$^*(n, k)$ if $X \neq \emptyset$,
- X is $m + 1$-dense$^*(n, k)$ if
  - for any coloring $P : [X]^n \to k$, there exists a homogeneous set $Y \subseteq X$ such that Y is m-dense$^*(n, k)$,
  - $\{x \in X \mid x > 2^{\min X}\}$ is m-dense$^*(n, k)$.

RT^n_k without Σ_1-induction

Then, we have the following.

Theorem

Let φ be a Π^0_2-sentence. If $\mathrm{WKL}^*_0 + \mathrm{RT}^n_k \vdash \varphi$, *then* $\mathrm{RCA}^*_0 \vdash \varphi$.

Recall that the Π^0_2 part of RCA^*_0 is EFA, *i.e.*, its provably recursive functions are elementary functions.
In general,

Theorem

Let T be a set of Π^0_2-formulas. Let φ be a Π^0_2-sentence. If $\mathrm{WKL}^*_0 + T + \mathrm{RT}^n_k \vdash \varphi$, *then* $\mathrm{RCA}^*_0 + T \vdash \varphi$.

RT^n_k without Σ_1-induction

Then, we have the following.

Theorem

Let φ be a Π^0_2-sentence. If $\mathrm{WKL}^*_0 + \mathrm{RT}^n_k \vdash \varphi$, then $\mathrm{RCA}^*_0 \vdash \varphi$.

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Let T be a set of Π^0_2-formulas. Let φ be a Π^0_2-sentence. If $\mathsf{WKL}^*_0 + T + \mathrm{RT}^n_k \vdash \varphi$, then $\mathsf{RCA}^*_0 + T \vdash \varphi$.

References

- Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. Journal of Symbolic Logic, 66(1):1–55, 2001.
- Chi-Tat Chong, Theodore A. Slaman, and Yue Yang. The metamathematics of Stable Ramsey's Theorem for pairs. to appear.
- Denis R. Hirschfeldt. Slicing the truth: On the computability theoretic and reverse mathematical analysis of combinatorial principles. to appear.
- J. B. Paris. Some independence results for Peano Arithmetic. Journal of Symbolic Logic, 43(4):725–731, 1978.
- Y. On the strength of Ramsey's theorem without Σ_1-induction. Mathematical Logic Quarterly 59(1-2), 108–111 (2013).

Outline

1. Computability vs NS models –on Ramsey's thm–
   - Formalizing Computability
   - Hybrid method Computability and NS models
   - Classical methods survive
2. Nonstandard analysis and RM
3. Some recent ideas
   - Random preserving extension
   - RM over RCA_0^*

JAF33, June 17, 2014

Nonstandard analysis and second-order arithmetic

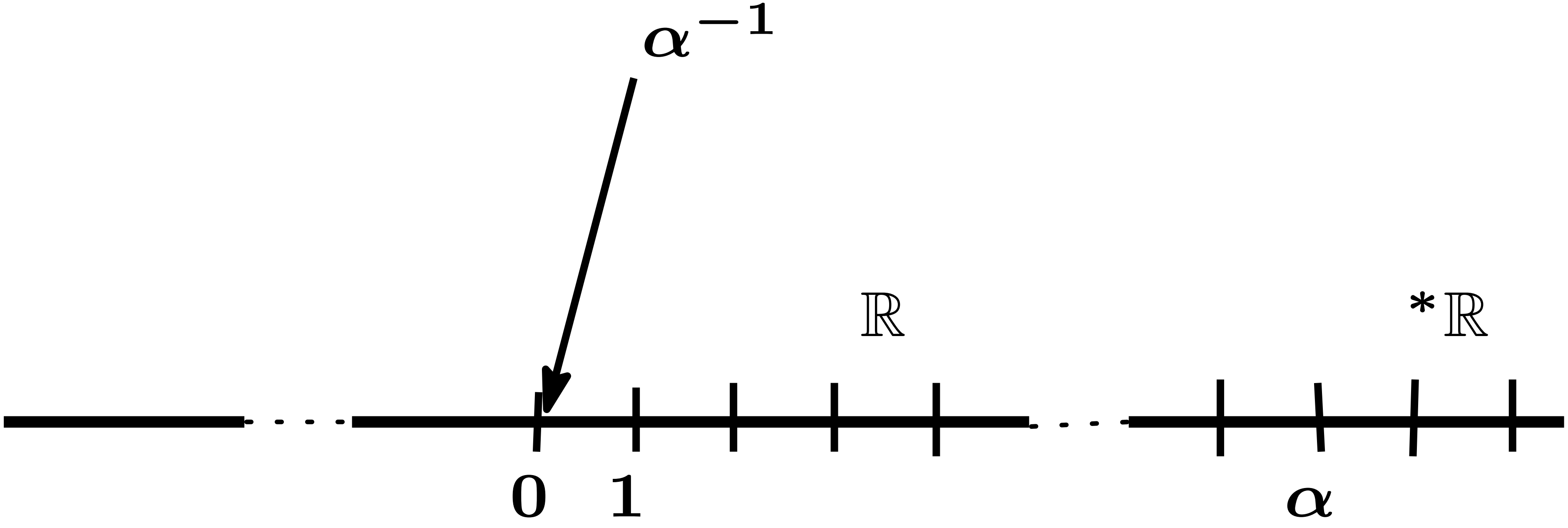
JAIST

Keita Yokoyama

Non-standard analysis

Non-standard analysis was introduced by Abraham Robinson in 1960s (based on model theory).

- Expanding the universe ($\mathbb{N} \subseteq \mathbb{N}^*$, $\mathbb{R} \subseteq \mathbb{R}^*$), we can use infinitesimals (inifinitely large and small numbers).



Non-standard analysis

Example. Let f be a continuous function, and f^* be a non-standard expansion of f. Let $\omega \in \mathbb{N}^* \setminus \mathbb{N}$ be a infinitely large number. Then, the Riemann integral and the derivative are defined as follows:

Riemann integral:

$$\int_0^1 f(x)\,dx = \mathrm{st}\left(\sum_{k=1}^{\omega} \frac{f^*(k/\omega)}{\omega}\right).$$

derivative:

$$f'(a) = \mathrm{st}\left(\frac{f^*(a+1/\omega) - f^*(a)}{1/\omega}\right).$$

Non-standard analysis

Example (Bolzano Weierstraß theorem).
Let $\langle a_n \mid n \in \mathbb{N} \rangle$ be a real sequence.
Let $\langle a_n^* \mid n \in \mathbb{N}^* \rangle$ be the non-standard expansion of $\langle a_n \mid n \in \mathbb{N} \rangle$.
Then, for any infinitely large number $\omega \in \mathbb{N}^* \setminus \mathbb{N}$, there exists a subsequence $\langle a_{n_i} \mid i \in \mathbb{N} \rangle$ which converges to $r := \mathrm{st}(a_\omega^*)$.

We can do mathematics only by using bounded formulas or less complicated $(\Sigma_1^0 \cup \Pi_1^0)$ formulas.

Non-standard analysis and RM

1. Model theoretic non-standard arguments

Within a countable model of $\mathbf{WKL_0}$ or $\mathbf{ACA_0}$, we can do non-standard analysis by means of weak saturation, standard part principle, . . .

- Non-standard arguments for $\mathbf{WKL_0}$ (Tanaka)
  - existence of Haar measure (Tanaka/Yamazaki)
- Non-standard arguments for $\mathbf{ACA_0}$
  - Riemann mapping theorem (Y)

Non-standard analysis and RM

1. Model theoretic non-standard arguments

2. Non-standard arithmetic

Big five systems are characterized by non-standard arithmetic (Keisler).

We combine 1 and 2, and introduce non-standard second order arithmetic.

3. Non-standard second order arithmetic

Non-standard analysis and RM

3. Non-standard second order arithmetic

1. Expansions of second order arithmetic and non-standard arithmetic.
2. We can do analysis in both ‘standard structure’ and ‘non-standard structure’.
3. We can use typical non-standard priciples such as ‘standard part priciple’, ‘transfer principle’,. . .
4. Conservation: if we prove a ‘standard theorem’ within a NS-system, then we can prove the same theorem within a corresponding (standard) second order arithmetic.

Non-standard analysis and RM

1. **Model theoretic non-standard arguments**
2. **Non-standard arithmetic**
3. **Non-standard second order arithmetic**

- We can do non-standard analysis in non-standard second order arithmetic.
- Using non-standard second order arithmetic and conservation, we can prove standard theorems in a weak second order arithmetic easily (original purpose).
- We can do Reverse Mathematics for non-standard analysis.

Language $\mathcal{L}_2^*$

Language of non-standard second order arithmetic ($\mathcal{L}_2^*$) are the following:

- s number variables: $x^{\mathrm{s}}, y^{\mathrm{s}}, \ldots$,
- $*$ number variables: $x^*, y^*, \ldots$,
- s set variables: $X^{\mathrm{s}}, Y^{\mathrm{s}}, \ldots$,
- $*$ set variables: $X^*, Y^*, \ldots$,
- s symbols: $\mathbf{0}^{\mathrm{s}}, \mathbf{1}^{\mathrm{s}}, =^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}}$,
- $*$ symbols: $\mathbf{0}^*, \mathbf{1}^*, =^*, +^*, \cdot^*, <^*, \in^*$,
- function symbol: $\surd$.

s-structure and $*$-structure

M^{s}: range of $x^{\mathrm{s}}, y^{\mathrm{s}}, \ldots$,
M^*: range of $x^*, y^*, \ldots$,
S^{s}: range of $X^{\mathrm{s}}, Y^{\mathrm{s}}, \ldots$,
S^*: range of $X^*, Y^*, \ldots$.

$V^{\mathrm{s}} = (M^{\mathrm{s}}, S^{\mathrm{s}}; 0^{\mathrm{s}}, 1^{\mathrm{s}}, \ldots)$: s-$\mathcal{L}_2$ structure.
$V^* = (M^*, S^*; 0^*, 1^*, \ldots)$: $*$-$\mathcal{L}_2$ structure.
$\surd : M^{\mathrm{s}} \cup S^{\mathrm{s}} \to M^* \cup S^*$: embedding.

We usually regard M^{s} as a subset of M^*.

(Notations)

Let φ be an $\mathcal{L}_2$-formula.

- φ^{s} : $\mathcal{L}_2^*$ formula constructed by adding $^{\mathrm{s}}$ to any $\mathcal{L}_2$ symbols in φ.
- φ^* : $\mathcal{L}_2^*$ formula constructed by adding * to any $\mathcal{L}_2$ symbols in φ.
- $\check{x}^{\mathrm{s}} := \sqrt{}(x^{\mathrm{s}})$.
- $\check{X}^{\mathrm{s}} := \sqrt{}(X^{\mathrm{s}})$.

We usually omit $^{\mathrm{s}}$ and * of relations $=, \leq, \in$.

We often say "φ holds in V^{s} (in V^*)" when φ^{s} (φ^*) holds.

Typical axioms of non-standard analysis

emb : "$\surd$ is an injective homomorphism".

e : $\forall x^* \forall y^{\mathrm{s}} (x^* < \check{y}^{\mathrm{s}} \to \exists z^{\mathrm{s}} (x^* = \check{z}^{\mathrm{s}}))$.

fst : $\forall X^* (\mathrm{card}(X^*) \in M^{\mathrm{s}}$

$\to \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}} (x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in X^*)$.

st : $\forall X^* \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}} (x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in X^*)$.

Σ^i_j**overspill(saturation) :**

$\forall x^* \forall X^* (\forall y^{\mathrm{s}} \exists z^{\mathrm{s}} (z^{\mathrm{s}} \geq y^{\mathrm{s}} \wedge \varphi(\check{z}^{\mathrm{s}}, x^*, X^*)^*)$

$\to \exists y^* (\forall w^{\mathrm{s}} (y^* > \check{w}^{\mathrm{s}}) \wedge \varphi(y^*, x^*, X^*)^*))$

for any $\Sigma^i_j(\mathcal{L}_2)$-formula $\varphi(z, x, X)$.

Typical axioms of non-standard analysis

$\Sigma^i_j\mathbf{equiv} : (\varphi^{\mathrm{s}} \leftrightarrow \varphi^*)$

for any $\Sigma^i_j(\mathcal{L}_2)$-sentence φ.

$\Sigma^i_j\mathbf{TP} : \quad \forall x^{\mathrm{s}} \forall X^{\mathrm{s}} (\varphi(x^{\mathrm{s}}, X^{\mathrm{s}})^{\mathrm{s}} \leftrightarrow \varphi(\check{x^{\mathrm{s}}}, \check{X^{\mathrm{s}}})^*)$

for any $\Sigma^i_j(\mathcal{L}_2)$-formula $\varphi(x, X)$.

$$\mathbf{LMP} : \quad \forall H^* \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}\ \forall T^* \subseteq 2^{<H^*}$$

$$\mathrm{st}\left(\frac{\mathrm{card}(\{\sigma^* \in T^* \mid \mathrm{lh}(\sigma^*) = H^*\})}{2^{H^*}}\right) > 0$$

$$\to \exists \sigma^* \in T^* \mathrm{lh}(\sigma^*) = H^* \wedge \sigma^* \cap \mathbb{N}^{\mathrm{s}} \in V^{\mathrm{s}}.$$

(An NS-tree which has a positive measure has a standard path.)

NS-systems

We define systems of non-standard second order arithmetic as follows.

$$\textbf{ns-BASIC} = (\textbf{RCA}_0)^{\mathrm{s}} + \mathbf{emb} + \mathbf{e} + \mathbf{fst} + \boldsymbol{\Sigma}^0_1\mathbf{overspill} + \boldsymbol{\Sigma}^1_2\mathbf{equiv} + \boldsymbol{\Sigma}^0_0\mathbf{TP}.$$

$$\textbf{ns-WKL}_0 = \textbf{ns-BASIC} + \mathbf{st}.$$

$$\textbf{ns-ACA}_0 = \textbf{ns-BASIC} + \mathbf{st} + \boldsymbol{\Sigma}^1_1\mathbf{TP}.$$

$$\textbf{ns-WWKL}_0 = \textbf{ns-BASIC} + \mathbf{LMP}.$$

Since $\mathbf{st}$ implies $\mathbf{LMP}$, we have

$$\textbf{ns-BASIC} < \textbf{ns-WWKL}_0 < \textbf{ns-WKL}_0 < \textbf{ns-ACA}_0.$$

$$\text{ns-BASIC} = (\mathsf{RCA}_0)^{\mathrm{s}} + \mathrm{emb} + \mathrm{e} + \mathrm{fst} + \Sigma^0_1\mathrm{overspill} + \Sigma^1_2\mathrm{equiv} + \Sigma^0_0\mathrm{TP}.$$

$$\Leftarrow M^{\mathrm{s}} \subsetneq_e M^* \text{ (semi-regurar)},$$

$$S^{\mathrm{s}} \subseteq \mathrm{Cod}(M^{\mathrm{s}}/M^*) = S^* \restriction M^{\mathrm{s}}$$

$$\text{ns-}\mathsf{WKL}_0 = \text{ns-BASIC} + \mathrm{st}.$$

$$\Leftarrow S^{\mathrm{s}} = \mathrm{Cod}(M^{\mathrm{s}}/M^*) = S^* \restriction M^{\mathrm{s}}$$

$$\text{ns-}\mathsf{ACA}_0 = \text{ns-BASIC} + \mathrm{st} + \Sigma^1_1\mathrm{TP}.$$

$$\Leftarrow (M^{\mathrm{s}}, S^{\mathrm{s}}) \prec_{\Sigma^1_1} (M^*, S^*)$$

$$\text{ns-}\mathsf{WWKL}_0 = \text{ns-BASIC} + \mathrm{LMP}.$$

$$\Leftarrow S^{\mathrm{s}} \subseteq_r \mathrm{Cod}(M^{\mathrm{s}}/M^*) = S^* \restriction M^{\mathrm{s}}$$

NS-systems

Theorem 1.

1. $\mathbf{ns\text{-}WKL_0} \vdash (\mathbf{WKL_0})^{\mathbf{s}} + (\mathbf{WKL_0})^*$.
2. $\mathbf{ns\text{-}WKL_0}$ is a conservative extension of $\mathbf{WKL_0}$.

Theorem 2.

1. $\mathbf{ns\text{-}ACA_0} \vdash (\mathbf{ACA_0})^{\mathbf{s}} + (\mathbf{ACA_0})^*$.
2. $\mathbf{ns\text{-}ACA_0}$ is a conservative extension of $\mathbf{ACA_0}$.

Theorem 3 (Simpson-Y)**.**

1. $\mathbf{ns\text{-}WWKL_0} \vdash (\mathbf{WWKL_0})^{\mathbf{s}} + (\mathbf{WWKL_0})^*$.
2. $\mathbf{ns\text{-}WWKL_0}$ is a conservative extension of $\mathbf{WWKL_0}$.

RM for non-standard analysis

Theorem 4. The following are equivalent over **ns-BASIC**.

1. **ns-WKL$_0$**.
2. For any continuous function f^{s} on $[0,1]$ in V^{s}, there exists a piecewise linear s-continuous continuous function f^* on $[0,1]$ in V^* such that $\mathrm{st}(f^*) = f^{\mathrm{s}}$.
3. For any totally bounded complete separable metric space $\langle A^{\mathrm{s}}, d^{\mathrm{s}} \rangle$ in V^{s}, there exist $A^* \supset A^{\mathrm{s}}$ and $d^* \supset d^{\mathrm{s}}$ in V^* such that
$$\hat{A}^* = \bigcup_{x^{\mathrm{s}} \in \hat{A}^{\mathrm{s}}} \mathrm{mon}(x^{\mathrm{s}}).$$
4. Non-standard Jordan curve theorem:
for any Jordan curve J^{s}, there exist non-standard arcwise connected disjoint open sets D_1^*, D_2^* such that $\partial D_1^* = \partial D_2^* = \mathbb{R}^{*2} \setminus D_1^* \cup D_2^*$ and $\mathrm{st}(\partial D_1^*) = J^{\mathrm{s}}$.

Theorem 5. The following are equivalent over **ns-BASIC**.

1. **ns-WWKL$_0$**.
2. $L(\mathrm{st}^{-1}(A^{\mathrm{s}})) \leq \alpha^{\mathrm{s}} \Leftrightarrow \mu(A^{\mathrm{s}}) \leq \alpha^{\mathrm{s}}$ for any $A^{\mathrm{s}} \subseteq [0,1]$, where $L(\mathrm{st}^{-1}(A^{\mathrm{s}})) = \inf\{L(B^*) \mid \mathrm{st}^{-1}(A^{\mathrm{s}}) \subseteq B^* \subseteq \Omega\}$.
3. If F^* is an s-bounded function on $[0,1]$, f^{s} is a pre-standard part of F^* and $H^* \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}}$, then f^{s} is integrable on $[0,1]$ and

$$\int_0^1 f^{\mathrm{s}}(x)dx = \mathrm{st}\left(\sum_{i \leq H^*} \frac{F^*(i/H^*)}{H^*}\right).$$

Theorem 6. The following are provable in **ns**-$\mathbf{ACA_0}$.
1. Transfer principle for real numbers.
2. Transfer principle for continuous functions.
3. Non-standard version of Bolzano/Weierstraß theorem.
4. Non-standard version of Ascoli's lemma.
5. Non-standard version of Reimann mapping theorem.

Remark that they are not equivalent to **ns**-$\mathbf{ACA_0}$ over **ns**-**BASIC**. On the other hand, Sam Sanders did some RM for $\mathbf{\Pi_1}$-transfer principle in a different framework.

Question 1.
Are they all equivalent to **ns**-$\mathbf{ACA_0}$ over **ns**-**BASIC** plus some basic notion?

We can apply these results to standard Reverse Mathematics.

Back to standard RM

Corollary 7. The following is equivalent over $\mathbf{RCA_0}$.

1. $\mathbf{WKL_0}$.
2. JRMT: for any Jordan curve J, there exists a biholomorphism h from $\Delta(0;1)$ to $D \subseteq \mathbb{C}$ such that $\partial D = J$.

Proof 1 $\rightarrow$ 2. By the conservation result, we only need to show **ns-$\mathbf{WKL_0}$** $\vdash$(JRMT)$^{\mathrm{s}}$.
By the previous theorem, there exists a non-standard biholomorphism h^* from $\Delta(0;1)$ to $D^* \subseteq \mathbb{C}^*$ such that $\mathrm{st}(\partial D^*) = J^{\mathrm{s}}$.
By the Schwarz lemma, $h^{*\prime}$ is bouded on $\Delta(1-2^{-i})$ for any $i \in \mathbb{N}^{\mathrm{s}}$. Thus, h^* is s-continuous on $\Delta(1)$.
Then we can easily show that $h^{\mathrm{s}} = \mathrm{st}(h^*)$ is a desired biholomorphic function in V^{s}.
Hence **ns-$\mathbf{WKL_0}$** $\vdash$(JRMT)$^{\mathrm{s}}$. □

NS-systems $\Rightarrow$ RM

1. Prove "simple version" of the target theorem within $\mathbf{RCA_0}$.
2. Use "non-standard approximation property" which is equivalent to **ns**-$\mathbf{WKL_0}$, then we get a $\mathbf{WKL_0}$ version of the target theorem.
3. Use "transfer principle" which is equivalent to **ns**-$\mathbf{ACA_0}$, then we get the full version of the target theorem within $\mathbf{ACA_0}$.

NS-systems $\Rightarrow$ RM

$\mathbf{RCA_0}$ version:

The following are provable within **$\mathbf{RCA_0}$**.

- Every polynomial on $[0, 1]$ has a maximum.
- For every polynomial f on $\Delta(1) \subseteq \mathbb{C}$, $|f|$ has a maximum on $\partial\Delta(1)$.
- Jordan curve theorem for a piecewise linear Jordan curve.
- Riemann mapping theorem for a polygonal region.

NS-systems $\Rightarrow$ RM

Use "non-standard approximation property":

The following are provable within $\mathsf{ns\text{-}WKL_0}$.

- Every continuous function on a Jordan region has a maximum.
- For every holomorphic function f on a Jordan region D, $|f|$ has a maximum on ∂D.
- Jordan curve theorem.
- Riemann mapping theorem for a Jordan region.

NS-systems $\Rightarrow$ RM

Use conservativity:

The following are provable within $\mathbf{WKL_0}$.

- Every continuous function on a Jordan region has a maximum.
- For every holomorphic function f on a Jordan region D, $|f|$ has a maximum on ∂D.
- Jordan curve theorem.
- Riemann mapping theorem for a Jordan region.

NS-systems $\Rightarrow$ RM

Use "transfer principle":

The following are provable within **ns-$\mathbf{ACA_0}$**.

- Every continuous function on a compact separable metric space has a maximum.
- For every holomorphic function f on a bounded closed set D, $|f|$ has a maximum on ∂D.
- For every normal family F_D and $z \in D$, $\max\{|f'(z)| : f \in F_D\}$ exists.
- Riemann mapping theorem.

NS-systems $\Rightarrow$ RM

Use conservativity:

The following are provable within $\mathbf{ACA_0}$.

- Every continuous function on a compact separable metric space has a maximum.
- For every holomorphic function f on a bounded closed set D, $|f|$ has a maximum on ∂D.
- For every normal family F_D and $z \in D$, $\max\{|f'(z)| : f \in F_D\}$ exists.
- Riemann mapping theorem.

NS-systems $\Rightarrow$ RM

1. Prove "simple version" of the target theorem within $\mathbf{RCA_0}$.
2. Use "non-standard approximation property" which is equivalent to **ns**-$\mathbf{WKL_0}$, then we get a $\mathbf{WKL_0}$ version of the target theorem.
3. Use "transfer principle" which is equivalent to **ns**-$\mathbf{ACA_0}$, then we get the full version of the target theorem within $\mathbf{ACA_0}$.

$\Rightarrow$ Given a simple version of a standard theorem in $\mathbf{RCA_0}$, we can get a $\mathbf{WWKL_0}$ version, a $\mathbf{WKL_0}$ version and an $\mathbf{ACA_0}$ version automatically.

H. Jerome Keisler, Nonstandard arithmetic and reverse mathematics, *The Bulletin of Symbolic Logic*, 12(1):100.125, 2006.

Kazuyuki Tanaka, The self-embedding theorem of $\mathbf{WKL_0}$ and a non-standard method, *Annals of Pure and Applied Logic*, 84:41-49, 1997.

Yoshihiro Horihata and Y, Nonstandard second-order arithmetic and Riemann's mapping theorem, *Annals of Pure and Applied Logic* 165, pages: 520–551, 2014.

Stephen G. Simpson and Y, A nonstandard counterpart of WWKL, *Notre Dame Journal of Formal Logic* 52(3), pages: 229–243, 2011.

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Differentiation theorem

In computable analysis, the following result is known.

Theorem (Demuth, 1975)

If $f : [0, 1] \to \mathbb{R}$ is a computable function with bounded variation and $z \in [0, 1]$ is Martin-Löf random, then f is differentiable at z.

By this result, one can conjecture the following.

Theorem ($WWKL_0$)

For any continuous function $f : [0, 1] \to \mathbb{R}$ with bounded variation, there exists $z \in [0, 1]$ such that f is (pseudo-)differentiable at z. In fact, f is differentiable almost everywhere. ((pseudo-)differentiable: the ratio of differences converges in the Cauchy sense, but the value might not exist.)

This is true, but formalizing the original proof is hard.

Differentiation theorem

In computable analysis, the following result is known.

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Differentiation theorem

We want to convince the following two results.

Theorem (Greenberg/Miller/Nies/Slaman)

Within WKL_0*, any continuous function* $f : [0,1] \to \mathbb{R}$ *is a difference of two monotone functions.*

Theorem (Brattka/Miller/Nies)

Any computable monotone function $f : [0,1] \to \mathbb{R}$ *is differentiable at any Martin-Löf random points.*
This is formalizable within RCA_0*, thus,* WWKL_0 *proves that any computable monotone function* $f : [0,1] \to \mathbb{R}$ *is differentiable almost everywhere.*

Extension with randomness preservation

Lemma (Simpson/Y)

For any countable model $(M, S) \models \mathrm{WWKL}_0$, there exists $\bar{S} \supseteq S$ such that $(M, \bar{S}) \models \mathrm{WKL}_0$ and the following holds:

(†) *for any $A \in \bar{S}$ there exists $B \in S$ such that B is Martin-Löf random relative to A.*

Idea of the proof.

If $f : [0, 1] \to \mathbb{R}$ is a continuous function in $(M, S) \models \mathrm{WWKL}_0$, then, f is a difference of two monotone functions $f = g - h$ in $(M, \bar{S})$. Take $z \in [0, 1] \cap S$ such that z is ML-random relative to $g \oplus h$, then f is differentiable at z.

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Reverse mathematics over RCA_0^*

(Over RCA_0^*)

The following are equivalent to RCA_0.

- Every finitely generated vector space has a basis.
- For every countable field K, every polynomial $f(x) \in K[x]$ has only finitely many roots in K.

The following are equivalent to WKL_0^*.

- Every countable ring has a prime ideal.
- Σ^0_1-determinacy in Cantor space.

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Question from second-order categoricity of $\mathbb{N}$

We want to characterize $\mathbb{N}$ by second-order categoricity within a system as weak as possible.

Question (Simpson/Y)

Is $\mathbb{N}$ second-order characterizable within RCA_0^*?
Precisely, we want a second-order statement φ such that

- RCA_0^* proves $\mathbb{N}$ satisfies φ.
- RCA_0^* proves the categoricity theorem for φ ($\mathrm{CT}(\varphi)$), where,
  - $\mathrm{CT}(\varphi)$: if an (inner) structure A satisfies φ then $A \cong \mathbb{N}$.

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Categoricity requires $\mathrm{I}\Sigma_1^0$

Theorem (Kołodziejczyk/Y)

Let φ be a second-order statement such that

WKL_0^* *proves* $\mathbb{N}$ *satisfies* φ.

Then, over RCA_0^*,

$\mathrm{CT}(\varphi)$ *implies* RCA_0.

This theorem is an easy consequence of the following.

Theorem (Kołodziejczyk/Y)

The following are equivalent over RCA_0^*.

1. $\neg\mathrm{I}\Sigma_1^0$.
2. *There exists an (inner) structure A for arithmetic, i.e., $A \subseteq \mathbb{N}, +_A \subseteq A \times A, \cdot_A \subseteq A \times A, \dots$, such that $|A| < |\mathbb{N}|$ and $(A, \{X \mid X \subseteq A\}) \models \mathrm{WKL}_0^*$.*

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(Key lemmas for the theorem)

Lemma

If (M,S) is a model of $\mathsf{RCA}_0^* + \neg\Sigma_1^0$-induction, then, there exists a Σ_1^0-definable cut $I \subseteq_e M$ such that I is closed under exponentiation.

Lemma

If (M,S) is a model of $\mathsf{RCA}_0^* + \neg\Sigma_1^0$-induction, and I is a Σ_1^0-definable cut, then there exist $A \in S$ and $f \in S$ such that f is an isomorphism $(A, \{X \in S \mid X \subseteq A\}) \cong (I, \{X \cap I \mid X \in S\})$.

Theorem (Simpson/Smith)

If (M,S) is a model of RCA_0^* and $I \subseteq_e M$ is a cut which is closed under exponentiation, then $(I, \{X \cap I \mid X \in S\}) \models \mathsf{WKL}_0^*$.

- Stephen G. Simpson and Rick L. Smith, Factorization of polynomials and Σ^0_1 induction, Annals of Pure and Applied Logic 31 (1986), 289–306.
- S. G. Simpson and Y, Reverse mathematics and Peano categoricity, Annals of Pure and Applied Logic 164 (2012), no. 3, 284–293.
- Leszek Aleksander Kołodziejczyk and Y, Categorical characterizations of the natural numbers require primitive recursion, preprint.

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